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Finite-Size Scaling for the 2D Ising Model with Minus Boundary Conditions

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We study the magnetization $m_L(h, \beta)$ for the Ising model on a large but finite lattice square under the minus boundary conditions. Using known large-deviation results evaluating the balance between the competing effects of the minus boundary conditions and the external magnetic field h, we describe the details of its dependence on h as exemplified by the finite-size rounding of the infinite-volume magnetization discontinuity and its shift with respect to the infinite-volume transition point.

KEY WORDS: Ising model; finite size scaling; first-order phase transitions.

1. INTRODUCTION

The ferromagnetic nearest-neighbour Ising model on $\mathbb{Z}^d$, $d \geqslant 2$, is perhaps the most familiar spin system undergoing a first-order phase transition. Its formal Hamiltonian is

$$H(\sigma) = -\sum_{\langle x, y \rangle} \sigma_x \sigma_y - h \sum_x \sigma_x, \tag{1.1}$$

where σ_x is the spin at the site $x \in \mathbb{Z}^d$ corresponding to the configuration $\sigma \in \{-1, 1\}^{\mathbb{Z}^d}$, $h \in \mathbb{R}$ is the external magnetic field, and $\langle x, y \rangle$ stands for a pair of nearest-neighbour sites x and y of $\mathbb{Z}^d$. The phase transition occurs at $h = 0$ whenever the inverse temperature β is sufficiently large: there exists a point $\beta_c < \infty$ such that, for inverse temperature $\beta > \beta_c$, the set of infinite-volume Gibbs states of the model at $h = 0$ contains two distinct

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pure phases (called the plus and the minus phase). Physically, the phase transition is characterized by the discontinuity of the specific magnetization $m(h, \beta) = -m(-h, \beta)$: whereas the one-sided limit[3] $m(0^+, \beta) := \lim_{h\to 0^+} m(h, \beta)$ equals zero for $0 < \beta < \beta_c$, it is positive once $\beta > \beta_c$, i.e. the *spontaneous magnetization* $m^* := m(0^+, \beta)$ appears at sub-critical temperatures. Equivalently, since the (specific Gibbs) free energy $f(h, \beta)$ is a concave function of h for any $\beta \geqslant 0$, it has one-sided partial derivatives $\frac{\partial f(h,\beta)}{\partial h^\pm}$ for all $\beta \geqslant 0$ and $h \in \mathbb{R}$, and these do not coincide if and only if $h = 0$ and $\beta > \beta_c$. Clearly, $-\frac{\partial f(h,\beta)}{\partial h^\pm} = m(h^\pm, \beta)$.

In general, any discontinuities that arise in a system exhibiting a first-order phase transition are smoothed out once the system is of a finite size. While the limiting free energy f (as well as its one-sided derivatives) does not depend on boundary conditions, its smoothed finite-volume version is heavily depending on particular boundary conditions. In refs. 2, 3, 5 and ref. 4 specific cases of periodic and free boundary conditions, respectively, were considered with a rather mild and well-controlled size dependence. Here we turn to the case of fixed (minus) boundary conditions. This is the case with a rather strong influence of the boundary conditions, and (as will be clarified later) one has to take into account the competing effects of boundary conditions and "long contours".

Let Λ_L be the square in $\mathbb{Z}^2$ centred at the origin whose side-length is $L \in \mathbb{N}$. In this paper we examine the ferromagnetic nearest-neighbour Ising model in Λ_L with the minus boundary conditions and an external field $h \in \mathbb{R}$ at a sub-critical temperature. Writing $\sigma_L\colon \Lambda_L \to \{-1, 1\}$ for a configuration in Λ_L, the corresponding *Hamiltonian under the fixed minus boundary conditions* is given by

$$H_{L,h}(\sigma_L) = -\sum_{\substack{\langle x,y\rangle:\\ x,y\in\Lambda_L}} \sigma_x\sigma_y + \sum_{\substack{\langle x,y\rangle:\\ x\in\Lambda_L,\, y\in\Lambda_L^c}} \sigma_x - h\,|\Lambda_L|\,S_L(\sigma_L). \tag{1.2}$$

Here

$$S_L(\sigma_L) := \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L} \sigma_x \tag{1.3}$$

is the *average spin* and $\Lambda_L^c := \mathbb{Z}^2 \setminus \Lambda_L$. The finite-volume *Gibbs measure* at the inverse temperature β associated with the Hamiltonian (1.2) is

$$\mu_{L,h}(\sigma_L) := \frac{e^{-\beta H_{L,h}(\sigma_L)}}{Z_{L,h}} \tag{1.4}$$

[3] This limit exists and is non-negative for all $\beta > 0$, see ref. 15, for instance.

with the *partition function* $Z_{L,h} := \sum_{\sigma_L \in \{-1,1\}^{\Lambda_L}} e^{-\beta H_{L,h}(\sigma_L)}$; in order to avoid heavy notation, we abstain here and hereafter from referring explicitly to the dependence of the various quantities on β, and we stress that we always take fixed minus boundary conditions. We shall use $\langle \cdot \rangle_{L,h}$ to denote the *expected value* with respect to $\mu_{L,h}$ and $P_{L,h}$ to denote the *distribution* of S_L under $\mu_{L,h}$.

Let $\beta > \beta_c$. If $h \neq 0$, boundary effects in the bulk of Λ_L disappear as Λ_L extends to the whole lattice $\mathbb{Z}^d$ because there is a unique Gibbs measure in the infinite volume. Nevertheless, the asymptotic behaviour of the Ising system may become rather delicate once we consider a magnetic field h_L which depends on L and decreases to zero as $L \to \infty$. This time, the boundary conditions could play an important role: while they force the system to be in the minus phase, a magnetic field $h_L \to 0^+$ draws it toward the plus phase. The situation when the influence of the magnetic field h_L (a bulk effect) is comparable to that of the minus boundary conditions (a surface effect) is of particular interest; this requires h_L to be of the order $1/L$. Therefore, it is natural to consider $h_L = B/L$, $B \in \mathbb{R}$. Schonmann and Shlosman[16] proved that there exists a unique point $B_0 = B_0(\beta) > 0$ such that $\mu_{L,B/L}$ converges weakly to the pure minus phase if $B < B_0$, while the limit is the pure plus phase if $B > B_0$. In both regimes, they investigated the exponential convergence of the average spin S_L under $\mu_{L,B/L}$ at the *surface* rate. To this end, they established a 'surface-order' large-deviation principle valid at $B = 0$, extending the results obtained by Ioffe.[11,12] Greenwood and Sun[10] pointed out (for any dimension $d \geqslant 2$) how the large-deviation principles with $B = 0$ and $B \neq 0$ are related, and inspected the surface-rate exponential convergence of S_L under $\mu_{L,B/L}$, too.

The basic picture behind these results is as follows. Let Λ_L be large but finite. If $B < B_0$, the minus boundary conditions prevail, selecting the minus phase in the box Λ_L, and S_L converges exponentially to $-m^* < 0$. If $B > B_0$, however, the magnetic field has the dominant effect, and the plus phase is outweighing in the system. This time, the average spin converges exponentially to a point $0 < m(B) < m^*$, c.f. (2.6), and a single droplet of the plus phase within Λ_L immersed into the minus phase is created. The most favourable shape of the droplet is not the usual equilibrium crystal (or Wulff) shape,[4] but rather its squeezed version (see ref. 16): whenever the droplet really appears, it necessarily touches the boundary of Λ_L along four equally long segments.

As a matter of fact, the droplet fluctuates around its deterministic Wulff shape. Accordingly, the macroscopic-scale separation of pure phases

[4] Roughly speaking, the Wulff shape is the one which minimizes the interfacial surface tension, assuming that its volume is fixed, see refs. 8, 16 for instance.

along the boundary of the equilibrium crystal shape is a subtle probabilistic problem. Its first rigorous study was done by Dobrushin, Kotecký, and Shlosman[8, 9] for the $2d$ Ising model at very low temperatures, using the cluster expansion analysis. The main part of their results was extended to all sub-critical temperatures in a non-perturbative approach of Ioffe and Schonmann.[13] In particular, they gave explicit asymptotics on the probabilities of the deviation of S_L from $-m^*$ at $h=0$ under the minus boundary conditions. For a recent review of main results of the rigorous microscopic theory of equilibrium crystal shapes, see ref. 1.

2. MAIN RESULT

For any $0<\vartheta<\infty$, let us consider the open interval

$$J_L(\vartheta) := \{h \in \mathbb{R} : |Lh - B_0| < \vartheta\}. \tag{2.1}$$

Our aim here is to examine, for any $\beta > \beta_c$, the asymptotic behaviour of the finite-volume specific magnetization

$$m_L(h, \beta) := \langle S_L \rangle_{L,h} = \frac{1}{\beta\, |\Lambda_L|} \frac{\partial}{\partial h} \log Z_{L,h} \tag{2.2}$$

and susceptibility

$$\chi_L(h, \beta) := \langle S_L^2 \rangle_{L,h} - (\langle S_L \rangle_{L,h})^2 = \frac{1}{(\beta\, |\Lambda_L|)^2} \frac{\partial^2}{\partial h^2} \log Z_{L,h} \tag{2.3}$$

on the interval $J_L(\vartheta)$ with $L\to\infty$. The resulting asymptotics presented in Theorem 2.2 reflects the mentioned balance between the competing influences of the magnetic field and the minus boundary conditions in our model. First, however, relying on results from ref. 16 and ref. 10, we explicitly show the limiting values with properly scaled external field, $h \sim 1/L$.

Proposition 2.1. Let $\beta>\beta_c$, $B\in\mathbb{R}$, and let $\{h_L\}$ be a sequence of real numbers such that $\lim_{L\to\infty} Lh_L = B$. Then the limit

$$\varphi(B) := \frac{1}{\beta} \lim_{L\to\infty} \frac{1}{L} \log \frac{Z_{L,h_L}}{Z_{L,0}} \tag{2.4}$$

exists and does not depend on the sequence h_L, it is a convex continuous function, and there exists a single point $B_0 = B_0(\beta) > 0$ at which φ is not differentiable. Moreover,

$$\lim_{L\to\infty} m_L(h_L, \beta) = \varphi'(B) \tag{2.5}$$

for every $B \neq B_0$ and φ' is explicitly given as

$$\varphi'(B) = \begin{cases} -m^* & \text{if} \quad B < B_0, \\ m(B) = m^* - \kappa/(2B)^2 & \text{if} \quad B > B_0, \end{cases} \tag{2.6}$$

with $\kappa = \kappa(\beta) \in (0, 4m^*(B_0)^2)$.

The first part of Proposition 2.1 readily follows from the Varadhan lemma[6, 7] and Theorem 1 from ref. 16, and can be found in ref. 10 for the special case $h_L = B/L$ with $B \geqslant 0$; the rest of the proposition is then easy to verify. We present the proof in the next section. It will turn out there that the point B_0 from this proposition coincides with the critical point B_0 of ref. 16 mentioned before—allowing thus to use the same symbol to denote it.

Let $B^\star := B_0(\frac{1}{2} + \frac{\kappa}{16m^*(B_0)^2})$. As $\kappa < 4m^*(B_0)^2$, one has $B^\star \in (B_0/2, 3B_0/4)$; it will be shown later that $m(B^\star) > -m^*$, see the remark after Theorem 3.2. Let us extend the function m defined on the interval (B_0, ∞) by (2.6). Namely, we take

$$m_+(B) := \begin{cases} m(B) & \text{for} \quad B \geqslant B^\star, \\ m(B^\star) & \text{for} \quad B \leqslant B^\star. \end{cases} \tag{2.7}$$

It is a continuous and non-decreasing function satisfying $m(B^\star) \leqslant m_+ < m^*$. Introducing the shorthands

$$\bar{m}(B) := \frac{m_+(B) + (-m^*)}{2}, \qquad \Delta m(B) := \frac{m_+(B) - (-m^*)}{2}, \tag{2.8}$$

and $\Delta := \Delta m(B_0) > 0$, we now formulate our main result.

Theorem 2.2. Let $\beta > \beta_c$, $0 < \vartheta < \infty$, and $0 < \delta < 1/4$. There exists $L_0 = L_0(\beta, \vartheta, \delta) < \infty$ such that for all $L > L_0$ the following is true.

(a) The susceptibility $\chi_L(h)$ attains its maximal value over the interval $J_L(\vartheta)$ at a unique point $h_\chi(L)$ (which does not depend on ϑ).

(b) The functions $R_L^{(0)}$, $R_L^{(1)}(h)$, and $R_L^{(2)}(h)$ defined by the equalities

$$h_\chi(L) = (B_0 + R_L^{(0)})/L, \tag{2.9}$$

$$m_L(h, \beta) = \bar{m}(Lh) + \Delta m(Lh) \tanh[\beta\Delta(h - h_\chi(L))\, L^2] + R_L^{(1)}(h), \tag{2.10}$$

and

$$\chi_L(h, \beta) = (\Delta m(Lh))^2 \cosh^{-2} [\beta \Delta (h - h_\chi(L)) L^2] + R_L^{(2)}(h), \qquad (2.11)$$

satisfy the following bounds:

$$|R_L^{(0)}| \leqslant 3(B_0)^3 L^{-\delta}/\kappa \qquad (2.12)$$

and

$$\sup_{h \in J_L(\vartheta)} |R_L^{(k)}(h)| \leqslant C L^{-\delta}, \qquad k = 1, 2, \qquad (2.13)$$

with a fixed finite constant C (not depending on β, ϑ, δ, and L).

We divide the proof of Theorem 2.2 into two parts. First, we prove a weaker version of the above theorem in which it is only claimed that $R_L^{(0)}$ as well as $\sup_{h \in J_L(\vartheta)} R_L^{(i)}(h)$, $i = 1, 2$, vanish as $L \to \infty$; this part is based on the large-deviation principle established in ref. 16 and it is the content of Section 3. In particular, results from ref. 16 yield explicit values for parameters B_0 and κ above. In order to obtain then the explicit bounds (2.12) and (2.13), we employ more accurate estimates from ref. 13, 1, 9 and Theorem 7.4.3 from ref. 17; this second step is presented in Section 4.

It should be pointed out that the division of the proof into two parts is not necessary and it could be carried out solely with the help of the above mentioned estimates. However, it seems to be more transparent to examine the problem by means of the large-deviation principle at the beginning and use the more precise information to estimate the error terms only afterwards. Moreover, once the class of models for which the surface-order large-deviation principles are established is extended (at present it only contains the two-dimensional Ising model), the first part of the proof will be readily applicable, yielding a result similar to Theorem 3.3 below (see ref. 14).

Finally, let us notice that, using more detailed analysis of the errors in the surface-order large deviations of the two-dimensional Ising model,[13, 1] one could expect that the upper bounds (2.12) and (2.13) may be improved to be of the order $L^{-1/4} \log^2 L$. However, one should not expect that, for $\sup_{h \in J_L(\vartheta)} |R_L^{(k)}(h)|$, $k = 1, 2$, an improvement of the order over $L^{-1/2}$ is possible. This follows from the fact that surface large-deviation rate function $\mathscr{W}_B(m)$ introduced below in (3.6) behaves (for $B \geqslant B_0$) like $(m - m(B))^2$ around its minimum at $m(B)$. This results in the bound (4.40) below, and inspecting it one could argue that, necessarily, $L^{1/2} \sup_{h \in J_L(\vartheta)} |R_L^{(k)}(h)| \to \infty$, $k = 1, 2$.

3. MAGNETIZATION AND THE LARGE-DEVIATION RATE FUNCTION

The aim of this section is to analyze the asymptotic behaviour (as $L \to \infty$) of the magnetization $m_L(h, \beta)$ and susceptibility $\chi_L(h, \beta)$ when $h \in J_L(\vartheta)$ and $\beta > \beta_c$, using the large-deviation principle and the related results from ref. 16, 10. First, let us fix some notation.

Definition 3.1. Let $I: \mathbb{R} \to [0, \infty]$ be a (lower semi-continuous) function with compact level sets[5], $I \not\equiv \infty$, and let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\lim_{n \to \infty} \varepsilon_n = 0$. We say that a sequence $\{P_n\}$ of probability measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, where $\mathscr{B}(\mathbb{R})$ is the Borel σ-field on $\mathbb{R}$, satisfies the *large-deviation principle* with the powers $\{\varepsilon_n\}$ and the rate I, and write $(P_n)^{\varepsilon_n} \to e^{-I}$, if

$$\sup_G e^{-I} \leqslant \varliminf_{n \to \infty} (P_n(G))^{\varepsilon_n} \qquad \text{for all} \quad G \subset \mathbb{R} \text{ open and} \tag{3.1}$$

$$\varlimsup_{n \to \infty} (P_n(F))^{\varepsilon_n} \leqslant \sup_F e^{-I} \qquad \text{for all} \quad F \subset \mathbb{R} \text{ closed.} \tag{3.2}$$

In the next theorem we gather up the results of Theorem 1 from ref. 16 and Theorem 3.3 from ref. 10. To this end, we introduce $\tau = \tau(\beta)$ and $w = w(\beta)$ to be the zero-field surface tension in the direction (0, 1) and the Wulff functional of the minimizing Wulff shape, respectively (see e.g. ref. 16 for precise definitions). They satisfy the relations $0 < 4\tau/3 < w < 4\tau$ for all $\beta > \beta_c$.

Theorem 3.2[(16, 10)]**.** Let $\beta > \beta_c$. Setting

$$\kappa := \frac{16\tau^2 - w^2}{2m^*} > 0 \tag{3.3}$$

and

$$B_0 := \frac{4\tau + w}{4m^*}, \tag{3.4}$$

we have:

[5] That is, the level sets $\operatorname{lev}_r(I) := \{x \in \mathbb{R} : I(x) \leqslant r\}$ are compact for all $r < \infty$. Such a function is automatically lower semi-continuous.

1. Let $m_t := -m^*(1-\frac{w^2}{8\tau^2}) \in (-m^*, m^*)$ and

$$\mathscr{W}_0(m) := \begin{cases} w\left(\dfrac{m+m^*}{2m^*}\right)^{1/2} & \text{if} \quad -m^* \leqslant m \leqslant m_t, \\ 4\tau - [\kappa(m^*-m)]^{1/2} & \text{if} \quad m_t \leqslant m \leqslant m^*, \\ \infty & \text{otherwise.} \end{cases} \tag{3.5}$$

Then $(P_{L,0})^{1/L} \to e^{-\beta \mathscr{W}_0}$.

2. Let

$$\mathscr{W}_B(m) := \mathscr{W}_0(m) - Bm + \mathscr{W}_0^*(B), \tag{3.6}$$

where $\mathscr{W}_0^*$ is the Legendre–Fenchel transform of $\mathscr{W}_0$. Let $\{h_L\}$, $h_L \in \mathbb{R}$, be a sequence satisfying $\lim_{L\to\infty} Lh_L = B \in \mathbb{R}$.[6] Then $(P_{L,h_L})^{1/L} \to e^{-\beta \mathscr{W}_B}$.

3. The derivative of the Legendre–Fenchel transform $\mathscr{W}_0^*$ of $\mathscr{W}_0$ has a unique discontinuity at B_0. Moreover, for all $B \neq B_0$, the equation $\mathscr{W}_B(m) = 0$ has a unique solution that equals the derivative $\frac{d\mathscr{W}_0^*}{dB}$ of $\mathscr{W}_0^*$, while for $B = B_0$ it has two solutions: $\frac{d\mathscr{W}_0^*}{dB+}$ and $\frac{d\mathscr{W}_0^*}{dB-}$.

Proposition 2.1 is a rather direct consequence of Theorem 3.2. In particular, it turns out that one can identify the function φ with $\mathscr{W}_0^*$ (this was anticipated by denoting B_0 the discontinuity point of both of them). Notice also that the constant $B^\star := B_0(\frac{1}{2} + \frac{\kappa}{16m^*(B_0)^2})$ defined in the previous section actually coincides with τ/m^*. Moreover, one has $m(B^\star) = m_t > -m^*$.

Proof of Proposition 2.1. Let $\beta > \beta_c$ and $B \in \mathbb{R}$. Consider the limit

$$\psi(B) = \frac{1}{\beta} \lim_{L\to\infty} \frac{1}{L} \log \langle e^{\beta BLS_L} \rangle_{L,0}. \tag{3.7}$$

In view of the Varadhan lemma[7] (refs. 6 and 7) and Theorem 3.2 (1), the limit exists and we get

$$\psi(B) = \frac{1}{\beta} \sup \{\beta Bx - \beta \mathscr{W}_0(x)\} = \mathscr{W}_0^*(B). \tag{3.8}$$

[6] In fact, Theorem 3.3 of ref. 10 only deals with $h_L = B/L$, where $B > 0$. It is clear, however, that the arguments used there work in our slightly more general case as well.

[7] Notice that $\lim_{M\to\infty} \overline{\lim}_{L\to\infty} \frac{1}{L} \log \sum_{m\in \mathrm{Ran}\, S_L:\, \beta Bm \geqslant M} e^{\beta BLm} P_{L,0}(m) = -\infty$ because the range $\mathrm{Ran}\, S_L$ of average spin S_L is bounded, $\mathrm{Ran}\, S_L \subset [-1, 1]$. This allows us to apply an extended version of Varadhan lemma,[(6)] Theorem 4.3.1.

With the help of (3.5), one may easily find[10] that

$$\psi(B) = \begin{cases} -m^*B & \text{if} \quad B \leqslant B_0, \\ m^*B - [4\tau - \kappa/(4B)] & \text{if} \quad B \geqslant B_0, \end{cases} \tag{3.9}$$

where $B_0 = (4\tau + w)/(4m^*)$. This point coincides with the critical point B_0 of ref. 16, see Theorem 2 stated therein.

We shall now show that the functions φ from (2.4) and ψ defined above actually coincide. Let thus $\{h_L\}$, $h_L \in \mathbb{R}$, be an arbitrary sequence such that $\lim_{L\to\infty} Lh_L = B$. Since

$$\frac{Z_{L,h_L}}{Z_{L,0}} = \sum_{\sigma_L \in \{-1,1\}^{\Lambda_L}} e^{\beta h_L |\Lambda_L| S_L(\sigma_L)} \mu_{L,0}(\sigma_L) = \langle e^{\beta h_L |\Lambda_L| S_L} \rangle_{L,0} \tag{3.10}$$

and the range of the average spin S_L is contained, by definition, in the interval $[-1, 1]$, we may evaluate

$$e^{-\beta |Lh_L - B| L} \langle e^{\beta BLS_L} \rangle_{L,0} \leqslant \langle e^{\beta h_L |\Lambda_L| S_L} \rangle_{L,0} \leqslant e^{\beta |Lh_L - B| L} \langle e^{\beta BLS_L} \rangle_{L,0}. \tag{3.11}$$

As a result,

$$\frac{1}{\beta} \lim_{L\to\infty} \frac{1}{L} \log \frac{Z_{L,h_L}}{Z_{L,0}} = \psi(B). \tag{3.12}$$

Thus, one has $\psi(B) = \varphi(B)$, and in order to verify (2.5), we notice that $m_L(h_L, \beta) = \langle S_L \rangle_{L,h_L} = \frac{1}{\beta L} \frac{d \log \langle e^{\beta BLS_L} \rangle_{L,0}}{dB} |_{B = Lh_L}$, getting the limit whenever the derivative of φ exists. ∎

The main result of the section is this simplified version of Theorem 2.2.

Theorem 3.3. Let $\beta > \beta_c$ and $0 < \vartheta < \infty$. There exists $L_0 = L_0(\beta, \vartheta) < \infty$ such that for all $L > L_0$ the claims a) and b) of Theorem 2.2 hold with (2.12) and (2.13) replaced by

$$\lim_{L\to\infty} R_L^{(0)} = 0 \qquad \text{and} \qquad \lim_{L\to\infty} \sup_{h \in J_L(\vartheta)} |R_L^{(k)}(h)| = 0, \qquad k = 1, 2, \tag{3.13}$$

respectively.

3.1. Proof of Theorem 3.3

Let $\beta > \beta_c$ and $L \in \mathbb{N}$. Given $h \in \mathbb{R}$ and a set $A \in \mathscr{B}(\mathbb{R})$ (which may depend on h) such that

$$Z_{L,h}(A) := \sum_{\substack{\sigma_L \in \Omega_L:\\ S_L(\sigma_L) \in A}} e^{-\beta H_{L,h}(\sigma_L)} > 0, \tag{3.14}$$

we define

$$\langle \cdot \mid A \rangle_{L,h} := \sum_{\substack{\sigma_L \in \Omega_L:\\ S_L(\sigma_L) \in A}} \cdot \frac{e^{-\beta H_{L,h}(\sigma_L)}}{Z_{L,h}(A)}. \tag{3.15}$$

In order to control the most relevant contributions to the partition function on $J_L(\vartheta)$, $\vartheta > 0$, we split—independently of L—the interval $J_L(\vartheta)$ into a *finite* number of disjoint sub-intervals as follows. Let $\epsilon \in (0, \epsilon_0(\vartheta))$ with

$$\epsilon_0(\vartheta) := 2 \min \{m(B_0+\vartheta) - m(B_0), m(B_0) - m(B^\star), \Delta m(B^\star)/4\} \tag{3.16}$$

and let us consider the sequence $\{m_i = m_0 + i\epsilon\}$, $i \in \mathbb{Z}$, where $m_0 = m_+(B_0) - \frac{\epsilon}{2}$. As the function m_+ is bounded, there clearly exist unique natural numbers $N_j = N_j(\beta, \vartheta, \epsilon)$, $j = 1, 2$, for which $m_+(B_0 - \vartheta) \in [m_{-N_1}, m_{-N_1+1})$ and $m_+(B_0+\vartheta) \in [m_{N_2}, m_{N_2+1})$. Let us consider now the sequence $B^{(i)}$ with $B^{(-N_1)} = B_0 - \vartheta$, $B^{(N_2+1)} = B_0 + \vartheta$, and $B^{(i)}$ for $i = -N_1+1, \ldots, N_1$ taken as the unique solution of the equation $m_+(B^{(i)}) = m_i$.[8] We split the interval $J_L(\vartheta)$,

$$J_L(\vartheta) = \bigcup_{i=-N_1}^{N_2} \mathscr{I}^{(\epsilon)}_{L,i}, \tag{3.17}$$

by taking

$$\mathscr{I}^{(\epsilon)}_{L,i} := \begin{cases} (B^{(i)}/L, B^{(i+1)}/L] & \text{if} \quad i = -N_1, \ldots, -1, \\ (B^{(0)}/L, B^{(1)}/L) & \text{for} \quad i = 0, \\ [B^{(i)}/L, B^{(i+1)}/L) & \text{if} \quad i = 1, \ldots, N_2. \end{cases} \tag{3.18}$$

Notice that, by definition, $B_0/L \in \mathscr{I}^{(\epsilon)}_{L,0}$. Moreover, introducing

$$\mathscr{C}^-(\epsilon) := (-m^* - \epsilon, -m^* + \epsilon), \tag{3.19}$$

[8] Since $\epsilon < \epsilon_0(\vartheta)$, it follows that $B^\star < B^{(0)} < B_0 < B^{(1)} < B_0 + \vartheta$.

and

$$\mathscr{C}^{+}(Lh, \epsilon) := (m_i - \epsilon, m_{i+1} + \epsilon) \qquad \text{for any} \quad h \in \mathscr{I}_{L,i}^{(\epsilon)}, \tag{3.20}$$

we have

$$|\langle S_L \mid \mathscr{C}^{+} \rangle_{L,h} - m_{+}(Lh)| \leqslant 2\epsilon \qquad \text{and} \qquad |\langle S_L \mid \mathscr{C}^{-} \rangle_{L,h} - (-m^*)| \leqslant \epsilon \tag{3.21}$$

for every $h \in J_L(\vartheta)$.

Taking $\mathscr{C}(Lh, \epsilon) := \mathscr{C}^{+}(Lh, \epsilon) \cup \mathscr{C}^{-}(\epsilon)$, we shall prove Theorem 3.3 with the help of the following sequence of lemmas.[9]

Lemma 3.4. Let $\beta > \beta_c$, $\vartheta > 0$, and $0 < \epsilon < \epsilon_0(\vartheta)$. For any $L > \epsilon^{-1/2}$ and $h \in J_L(\vartheta)$, we have

$$|\langle S_L \rangle_{L,h} - T(\phi^{(\epsilon)}(h); \bar{m}(Lh), \Delta m(Lh))| \leqslant 2 P_{L,h}(\mathscr{C}^c) + 3\epsilon. \tag{3.22}$$

Here

$$T(x; a, b) := a + b \tanh x, \qquad x, a, b \in \mathbb{R}, \tag{3.23}$$

and

$$\phi^{(L,\epsilon)}(h) := \frac{1}{2} \log \frac{Z_{L,h}(\mathscr{C}^{+})}{Z_{L,h}(\mathscr{C}^{-})} = \frac{1}{2} \log \frac{P_{L,h}(\mathscr{C}^{+})}{P_{L,h}(\mathscr{C}^{-})}. \tag{3.24}$$

Proof. Let $\beta > \beta_c$, $\vartheta > 0$, $\epsilon < \epsilon_0(\vartheta)$, and $L > \epsilon^{-1/2}$ be given. Let $h \in J_L(\vartheta)$ be arbitrary. Evidently,

$$\begin{aligned} \langle S_L \rangle_{L,h} &= \langle S_L \mid \mathscr{C} \rangle_{L,h} P_{L,h}(\mathscr{C}) + \langle S_L \mid \mathscr{C}^c \rangle_{L,h} P_{L,h}(\mathscr{C}^c) & (3.25)\\ &= \langle S_L \mid \mathscr{C} \rangle_{L,h} + (\langle S_L \mid \mathscr{C}^c \rangle_{L,h} - \langle S_L \mid \mathscr{C} \rangle_{L,h}) P_{L,h}(\mathscr{C}^c). & (3.26) \end{aligned}$$

Thus, using that $|S_L| \leqslant 1$, one has

$$|\langle S_L \rangle_{L,h} - \langle S_L \mid \mathscr{C} \rangle_{L,h}| \leqslant 2 P_{L,h}(\mathscr{C}^c). \tag{3.27}$$

[9] The fact that $\mathscr{C}^{+}$ and $\mathscr{C}^{-}$ are open is not important: the arguments of the proof also work if these are closed or half-open.

Observing that $\mathscr{C}^+ \cap \mathscr{C}^- = \varnothing$ (since $\epsilon < \Delta m(B^\star)/2$) and that $Z_{L,h}(\mathscr{C}^\pm) > 0$ (since $L > \epsilon^{-1/2}$), we readily get

$$\begin{aligned}\langle S_L \mid \mathscr{C} \rangle_{L,h} &= \frac{\langle S_L \mid \mathscr{C}^+ \rangle_{L,h} Z_{L,h}(\mathscr{C}^+) + \langle S_L \mid \mathscr{C}^- \rangle_{L,h} Z_{L,h}(\mathscr{C}^-)}{Z_{L,h}(\mathscr{C}^+) + Z_{L,h}(\mathscr{C}^-)} \\ &= \frac{\langle S_L \mid \mathscr{C}^+ \rangle_{L,h} + \langle S_L \mid \mathscr{C}^- \rangle_{L,h}}{2} \\ &\quad + \frac{\langle S_L \mid \mathscr{C}^+ \rangle_{L,h} - \langle S_L \mid \mathscr{C}^- \rangle_{L,h}}{2} \tanh(\phi^{(L,E)}(h)). \end{aligned} \tag{3.28}$$

Since $|\tanh x| \leqslant 1$ for all $x \in \mathbb{R}$, in view of (3.21) it follows that

$$|\langle S_L \mid \mathscr{C} \rangle_{L,h} - T(\phi^{(\epsilon)}(h); \bar{m}(Lh), \Delta m(Lh))| \leqslant 3\epsilon. \tag{3.29}$$

Combined with (3.27), we obtain the lemma. ∎

The next lemma provides bounds on the derivatives of $\langle S_L \rangle_{L,h}$ analogous to that from (3.22). To this end, we start with the following definition.

Definition 3.5. Let $k \in \mathbb{N}$. Given a set $\{g_1, ..., g_k\}$ of k real-valued functions in $\mathbb{R}$, we introduce

$$\mathscr{F}_k(\{g_j\}) := \sum_{j=1}^{k} (-1)^{j-1} (j-1)! \sum_{\{I_1, ..., I_j\}} \prod_{\ell=1}^{j} g_{|I_\ell|}, \tag{3.30}$$

where the second sum runs over all partitions $\{I_1, ..., I_j\}$, $j = 1, ..., k$, of the set $\{1, ..., k\}$ and $|I_\ell|$, $\ell = 1..., j$, is the cardinality of I_ℓ.

Lemma 3.6. There exist finite constants C_k, K_k, $k \in \mathbb{N}$, such that if $\beta > \beta_c$, $\vartheta > 0$, $0 < \epsilon < \epsilon_0(\vartheta)$, and $L > \epsilon^{-1/2}$, then

$$\left| \frac{1}{(\beta\,|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L \rangle_{L,h} - \mathscr{F}_k(\{T(\phi^{(L,E)}(h); \xi_j^+, \xi_j^-)\}) \right| \leqslant C_k\, P_{L,h}(\mathscr{C}^c) + K_k\, \epsilon \tag{3.31}$$

for all $h \in J_L(\vartheta)$. Here

$$\xi_j^\pm := \frac{(m_+(Lh))^j \pm (-m^*)^j}{2}, \qquad j = 1, ..., k. \tag{3.32}$$

Proof. In order to prove (3.31), it suffices—by Lemma A.2—to show that for any $k \in \mathbb{N}$ there exists a finite positive constant K_k such that

$$|\mathscr{F}_k(\{\langle (S_L)^j \mid \mathscr{C}\rangle_{L,h}\}) - \mathscr{F}_k(\{T(\phi^{(L,E)}(h); \xi_j^+, \xi_j^-)\})| \leqslant K_k \epsilon. \tag{3.33}$$

Indeed, taking into account (3.15) and (3.19) (cf. (3.21)), we have

$$\begin{aligned}
|\langle (S_L)^n \mid \mathscr{C}^+\rangle_{L,h} - (m_+(Lh))^n| &\leqslant \sum_{\substack{\sigma_L \in \Omega_L: \\ S_L(\sigma_L) \in \mathscr{C}^+}} |(S_L(\sigma_L))^n - (m_+(Lh))^n| \, \frac{e^{-\beta H_{L,h}(\sigma_L)}}{Z_{L,h}(\mathscr{C}^+)} \\
&\leqslant \sum_{\substack{\sigma_L \in \Omega_L: \\ S_L(\sigma_L) \in \mathscr{C}^+}} |S_L(\sigma_L) - m_+(Lh)| \sum_{r=0}^{n-1} |S_L(\sigma_L)|^r \, |m_+(Lh)|^{n-r-1} \, \frac{e^{-\beta H_{L,h}(\sigma_L)}}{Z_{L,h}(\mathscr{C}^+)} \\
&\leqslant \tilde{K}_n \epsilon
\end{aligned} \tag{3.34}$$

for all $n \in \mathbb{N}$ and $\tilde{K}_n = 2n$. Similarly, one has $|\langle (S_L)^n \mid \mathscr{C}^-\rangle - (-m^*)^n| \leqslant \tilde{K}_n \epsilon$ for all $n \in \mathbb{N}$ with the same constant $\tilde{K}_n$. Combined with the equality

$$\langle (S_L)^n \mid \mathscr{C}\rangle_{L,h} = \frac{\langle (S_L)^n \mid \mathscr{C}^+\rangle_{L,h} \, Z_{L,h}(\mathscr{C}^+) + \langle (S_L)^n \mid \mathscr{C}^-\rangle_{L,h} \, Z_{L,h}(\mathscr{C}^-)}{Z_{L,h}(\mathscr{C}^+) + Z_{L,h}(\mathscr{C}^-)}, \qquad n \in \mathbb{N}, \tag{3.35}$$

valid whenever $\epsilon < \Delta m(B^\star)/2$ and $L > \epsilon^{-1/2}$, referring to (3.28) it follows that

$$|\langle (S_L)^n \mid \mathscr{C}\rangle_{L,h} - T(\phi^{(L,E)}(h); \xi_n^+, \xi_n^-)| \leqslant \tilde{K}_n \epsilon \tag{3.36}$$

for all $n \in \mathbb{N}$.

Let now $k \in \mathbb{N}$. For any $j = 1, \ldots, k$ and any partition $\{I_1, \ldots, I_j\}$ one gets

$$\begin{aligned}
\prod_{\ell=1}^{j} \langle (S_L)^{|I_\ell|} \mid \mathscr{C}\rangle_{L,h} &= \prod_{\ell=1}^{j} T(\phi^{(L,E)}(h); \xi_{|I_\ell|}^+, \xi_{|I_\ell|}^-) \\
&\quad + \sum_{\substack{X \subset \{1,\ldots,j\}: \\ X \neq \{1,\ldots,j\}}} \prod_{r \in X} T(\phi^{(L,E)}(h); \xi_{|I_r|}^+, \xi_{|I_r|}^-) \\
&\quad \times \prod_{s \in \{1,\ldots,j\} \setminus X} (\langle (S_L)^{|I_s|} \mid \mathscr{C}\rangle_{L,h} - T(\phi^{(L,E)}(h); \xi_{|I_s|}^+, \xi_{|I_s|}^-)).
\end{aligned} \tag{3.37}$$

By virtue of (3.36), the obvious bound $|\xi_n^{\pm}| \leqslant 1$ for any $n \in \mathbb{N}$, and the fact that $\epsilon \leqslant \Delta m(B^\star)/2 \leqslant 1$, we arrive at (3.33). ▮

Next, let us examine the behaviour of the function $\phi^{(L,\epsilon)}(h)$ defined by (3.24).

Lemma 3.7. Let $\beta > \beta_c$, $\vartheta > 0$, and $0 < \epsilon < \epsilon_0(\vartheta)$.

(a) Let $L > \epsilon^{-1/2}$. Then the function $\phi^{(L,\epsilon)}(h)$ is finite on $J_L(\vartheta)$. In addition, it is analytic and increasing on $\mathscr{I}_{L,i}^{(\epsilon)}$ for each $i = -N_1, \ldots, N_2$.

(b) There exists a constant $L_1 = L_1(\beta, \vartheta, \epsilon) < \infty$, $L_1 \geqslant \epsilon^{-1/2}$, such that for $L > L_1$ the function $\phi^{(L,\epsilon)}(h)$ vanishes inside $J_L(\vartheta)$ at a unique point $h_0(L,\epsilon)$. Moreover, the limit $\lim_{L\to\infty} Lh_0(L,\epsilon)$ exists, it is independent of ϵ, and[10]

$$\lim_{L\to\infty} Lh_0(L,\epsilon) = B_0. \tag{3.38}$$

(c) Let[11]

$$\omega(h) := \beta\,(h - h_0(L,\epsilon))\,|\Lambda_L|\,\Delta. \tag{3.39}$$

There exist finite constants $L_2 = L_2(\beta, \vartheta, \epsilon)$, M_k, $k \in \mathbb{N}$, such that if $L > L_2$, then

$$\begin{aligned} &|\mathscr{F}_k(\{T(\phi^{(L,\epsilon)}(h);\, \xi_j^+, \xi_j^-)\}) - \mathscr{F}_k(\{T(\omega(h);\, \xi_j^+, \xi_j^-)\})| \\ &\quad \leqslant M_k\,(|\Delta m(Lh_0(L,\epsilon) + L^{-1/2}) - \Delta| + \epsilon)/\Delta \end{aligned} \tag{3.40}$$

for all $h \in J_L(\vartheta)$ and $k \in \mathbb{N}$.

Proof. Let $\beta > \beta_c$, $\vartheta > 0$, and $0 < \epsilon < \epsilon_0(\vartheta)$.

(a) Let $L > \epsilon^{-1/2}$. First, if $h \in J_L(\vartheta)$, then $Z_{L,h}(\mathscr{C}^{\pm}) > 0$ (since $L > \epsilon^{-1/2}$), and $\phi^{(L,\epsilon)}(h)$ is finite by its very definition (3.24). Second, let $i = -N_1, \ldots, N_2$ and let us consider the interval $\mathscr{I}_{L,i}^{(\epsilon)}$. By its definition, the set $\mathscr{C}^+$ is independent of h on $\mathscr{I}_{L,i}^{(\epsilon)}$, and, hence, the function $Z_{L,h}(\mathscr{C}^+)$ is

[10] Notice that (3.38) implies $(P_{L,h_0(L,\epsilon)})^{1/L} \to e^{-\beta \mathscr{W}_{B_0}}$ by Theorem 3.2 (2), where $\mathscr{W}_{B_0}$ is given by (3.6).

[11] Recall that $\Delta = \Delta m(B_0)$.

analytic in h on $\mathscr{I}^{(\epsilon)}_{L,i}$. The same is thus true for the function $\phi^{(L,\epsilon)}$. Moreover, for any h in the interior of $\mathscr{I}^{(\epsilon)}_{L,i}$ one has

$$\frac{1}{\beta\,|\Lambda_L|}\frac{\partial\phi^{(L,\epsilon)}(h)}{\partial h}=\frac{\langle S_L\,|\,\mathscr{C}^{\pm}\rangle_{L,h}-\langle S_L\,|\,\mathscr{C}^{-}\rangle_{L,h}}{2}\geqslant\frac{m_+(Lh)+m^*}{2}-\frac{3}{2}\epsilon>\frac{\epsilon_0(\vartheta)}{2} \tag{3.41}$$

with the help of (3.21) . Thus, the function $\phi^{(L,\epsilon)}$ is increasing on $\mathscr{I}^{(\epsilon)}_{L,i}$.

(b) Let $i=-N_1,\ldots,-1$. Then $(P_{L,B^{(i+1)}/L})^{1/L}\to e^{-\beta\mathscr{W}_{B^{(i+1)}}}$ in view of Theorem 3.2 (2). Observing further that, due to Theorem 3.2 (3), one has $\inf_{m\in\mathscr{C}^-}\mathscr{W}_B(m)=0$ whenever $B<B_0$, we get

$$\lim_{L\to\infty}\frac{1}{L}\,\phi^{(L,\epsilon)}(B^{(i+1)}/L)=-\frac{\beta}{2}\inf_{x\in[m_i-\epsilon,\,m_{i+1}+\epsilon]}\mathscr{W}_{B^{(i+1)}}(x)=-\frac{\beta}{2}\,\mathscr{W}_{B^{(i+1)}}(x_i) \tag{3.42}$$

for some $x_i\in[m_i-\epsilon,m_{i+1}+\epsilon]$. Taking into account the bounds

$$m_i-\epsilon>m_+(B_0-\vartheta)-2\epsilon\geqslant-m^*+4\epsilon_0(\vartheta)-2\epsilon>-m^*+2\epsilon_0(\vartheta) \tag{3.43}$$

as well as the fact that $B^{(i+1)}<B_0$, we can use Theorem 3.2 (3) once more to get $\mathscr{W}_{B^{(i+1)}}(x)>0$ and thus

$$\phi^{(L,\epsilon)}(B^{(i+1)}/L)<-\beta L\,\mathscr{W}_{B^{(i+1)}}(x_i)/4<0 \tag{3.44}$$

once L is sufficiently large (depending on β, ϑ, ϵ, and i). In a similar way,

$$\phi^{(L,\epsilon)}(B^{(i)}/L)>\beta L\,\mathscr{W}_{B^{(i)}}(x_i)/4>0 \tag{3.45}$$

for any $i=1,\ldots,N_2$ and some $x_i\in[m^*-\epsilon,m^*+\epsilon]$ once L is sufficiently large (depending on β, ϑ, ϵ, and i). Referring to the fact that $\phi^{(L,\epsilon)}$ is increasing on $\mathscr{I}^{(\epsilon)}_{L,i}$ for every $i=-N_1,\ldots,N_2$ by the claim (a) of this lemma and that N_1+N_2 is finite, one concludes that $\phi^{(L,\epsilon)}(h)\neq 0$ for all $h\in J_L(\vartheta)\setminus\mathscr{I}^{(\epsilon)}_{L,0}$ as soon as L is large enough (depending on β, ϑ, and ϵ).

Let $h\in\mathscr{I}^{(\epsilon)}_{L,0}$ now. According to the mean-value theorem,

$$\phi^{(L,\epsilon)}(h)=\phi^{(L,\epsilon)}(B_0/L)+(h-B_0/L)\,\frac{\partial\phi^{(L,\epsilon)}(\bar h)}{\partial h} \tag{3.46}$$

for some $\bar h$ between h and B_0/L. Taking into account that $\mathscr{W}_{B_0}(m_+(B_0))=0$ and thus $\inf_{m\in\mathscr{C}^+}\mathscr{W}_{B_0}(m)=0$, we get

$$\lim_{L\to\infty}\frac{\phi^{(L,\epsilon)}(B_0/L)}{L}=\frac{1}{2}\lim_{L\to\infty}\frac{1}{L}\log\frac{P_{L,B_0/L}(\mathscr{C}^+)}{P_{L,B_0/L}(\mathscr{C}^-)}=0 \tag{3.47}$$

according to Theorem 3.2 (3). With the help of (3.46) we thus get

$$\phi^{(L,\epsilon)}(h) \leqslant \phi^{(L,\epsilon)}(B_0/L) - \frac{B_0 - B^{(0)}}{2L} \frac{\beta\,|\Lambda_L|\,\epsilon_0(\vartheta)}{2}$$

$$< L\left(\frac{\phi^{(L,\epsilon)}(B_0/L)}{L} - \frac{\beta}{4}(B_0 - B^{(0)})\,\epsilon_0(\vartheta)\right) < 0 \qquad (3.48)$$

for any $h \in \mathscr{I}^{(\epsilon)}_{L,0}$ such that $h \leqslant (B_0 + B^{(0)})/(2L)$ once L is large enough (depending on β, ϑ, and ϵ). Analogously, one proves that $\phi^{(L,\epsilon)}(h) > 0$ for any $h \in \mathscr{I}^{(\epsilon)}_{L,0}$ such that $h \geqslant (B_0 + B^{(1)})/(2L)$ if L is sufficiently large (depending on β, ϑ, and ϵ). Since $\phi^{(L,\epsilon)}$ is continuous (it is analytic) and increasing on $\mathscr{I}^{(\epsilon)}_{L,0}$ for $L > \epsilon^{-1/2}$, this means that, for L large (depending on β, ϑ, and ϵ), a unique point $h_0(L,\epsilon)$ at which $\phi^{(L,\epsilon)}(h_0(L,\epsilon)) = 0$ exists, and

$$(B_0 + B^{(0)})/2 < L h_0(L,\epsilon) < (B_0 + B^{(1)})/2. \qquad (3.49)$$

Moreover, the relation (3.46) with $h = h_0(L,\epsilon)$ combined with (3.41) and (3.47) readily implies (3.38).

(c) Let $i = -N_1, \ldots, N_2$ be such that $i \neq 0$ and let $h_i \in \mathscr{I}^{(\epsilon)}_{L,i}$. Using that $\phi^{(L,\epsilon)}$ is increasing on $\mathscr{I}^{(\epsilon)}_{L,i}$ for $L > L_1$ and recalling the bounds (3.44) and (3.45) valid for $L > L_1$, we get $|\phi^{(L,\epsilon)}(h_i)| \geqslant \beta L\,\alpha_i(\beta,\vartheta,\epsilon)/4$ for $L > L_1$, where $\alpha_i > 0$ stands for $\mathscr{W}_{B^{(i+1)}}(x_i)$ if $i \leqslant -1$ or $\mathscr{W}_{B^{(i)}}(x_i)$ if $i \geqslant 1$. Moreover,

$$|\omega(h_i)| \geqslant \beta \min\{L h_0(L,\epsilon) - B^{(0)}, B^{(1)} - L h_0(L,\epsilon)\}\, L\Delta$$

$$> \beta \min\{B_0 - B^{(0)}, B^{(1)} - B_0\}\, \frac{L}{2}\, \Delta \qquad (3.50)$$

for $L > L_1$ by (3.49). Hence, observing that $1 - 2e^{-2|x|} \leqslant \tanh|x| \leqslant 1$, we have

$$|\tanh(\phi^{(L,\epsilon)}(h_i)) - \tanh(\omega(h_i))| \leqslant 2e^{-2\min\{|\phi^{(L,\epsilon)}(h_i)|,\,|\omega(h_i)|\}} \leqslant 2e^{-\beta L\,\alpha(\beta,\vartheta,\epsilon)/2} \qquad (3.51)$$

for L larger than some $\tilde{L}_2 = \tilde{L}_2(\beta,\vartheta,\epsilon)$, $\tilde{L}_2 \geqslant L_1$, with

$$\alpha(\beta,\vartheta,\epsilon) := \min\{\min_{\substack{-N_1 \leqslant i \leqslant N_2 \\ i \neq 0}} \alpha_i(\beta,\vartheta,\epsilon),\, 2\min\{B_0 - B^{(0)}, B^{(1)} - B_0\}\,\Delta\} > 0. \qquad (3.52)$$

Now, let us consider $h \in \mathscr{I}^{(\epsilon)}_{L,0}$ and take L so large that $L \geqslant \tilde{L}_2$ and

$$\mathfrak{I}_L := \{h' \in \mathbb{R}: |h' - h_0(L,\epsilon)|\, L \leqslant L^{-1/2}\} \subset \mathscr{I}^{(\epsilon)}_{L,0}. \qquad (3.53)$$

If $h \in \mathscr{I}^{(\epsilon)}_{L,0} \setminus \mathfrak{I}_L$, then $|\omega(h)| \geqslant \beta\Delta L^{1/2}$ and

$$|\phi^{(L,\epsilon)}(h)| = |h - h_0(L,\epsilon)| \, \frac{\partial\phi^{(L,\epsilon)}(\hat{h})}{\partial h} \geqslant \beta\epsilon_0 L^{1/2}/2 \tag{3.54}$$

according to the mean-value theorem and (3.41) (here $\hat{h}$ is some point between h and $h_0(L)$). Consequently,

$$|\tanh(\phi^{(L,\epsilon)}(h)) - \tanh(\omega(h))| \leqslant 2e^{-2\min\{|\phi^{(L,\epsilon)}(h)|, |\omega(h)|\}} \leqslant 2e^{-\beta\epsilon_0 L^{1/2}}. \tag{3.55}$$

On the other hand, if $h \in \mathfrak{I}_L$, then Lemma A.3, an upper bound similar to (3.41) (c.f. (3.21)), and the mean-value theorem yield

$$\begin{aligned}|\tanh(\phi^{(L,\epsilon)}(h)) - \tanh(\omega(h))| &\leqslant \frac{|\tanh(\omega(h))|}{|\omega(h)|} \, |\phi^{(L,\epsilon)}(h) - \omega(h)| \\ &\leqslant \frac{1}{|\omega(h)|} \, \beta \, |h - h_0(L,\epsilon)| \, |\Lambda_L| \left| \frac{1}{\beta\,|\Lambda_L|} \frac{\partial\phi^{(L,\epsilon)}(\hat{h})}{\partial h} - \Delta \right| \\ &\leqslant \frac{1}{\Delta}\left(|\Delta m(Lh) - \Delta| + \frac{3}{2}\epsilon \right) \leqslant \frac{1}{\Delta}\left(|\Delta m(Lh_0(L,\epsilon) + L^{-1/2}) - \Delta| + \frac{3}{2}\epsilon \right).\end{aligned} \tag{3.56}$$

Combined with (3.51), (3.55) (with the right hand side bounded by $\frac{\epsilon}{4\Delta}$ for L sufficiently large), and the obvious bound

$$|T(\phi^{(L,\epsilon)}(h); \xi_j^+, \xi_j^-) - T(\omega(h); \xi_j^+, \xi_j^-)| \leqslant |\xi_j^-| \, |\tanh(\phi^{(L,\epsilon)}(h)) - \tanh(\omega(h))|, \tag{3.57}$$

where $j \in \mathbb{N}$, we arrive at (3.40) with $k = 1$ and $M_1 = 2$ when we recall that $|\xi_j^{\pm}| \leqslant 1$ and realize that $\mathscr{F}_1(\{g_1\}) = g_1$. In addition, similarly to (3.37), one has

$$\begin{aligned}\prod_{\ell=1}^{j} T(\phi^{(L,\epsilon)}(h); \xi^+_{|I_\ell|}, \xi^-_{|I_\ell|}) &= \prod_{\ell=1}^{j} T(\omega(h); \xi^+_{|I_\ell|}, \xi^-_{|I_\ell|}) \\ &+ \sum_{\substack{X \subset \{1,\dots,j\}:\\ X \neq \{1,\dots,j\}}} \prod_{r \in X} T(\omega(h); \xi^+_{|I_r|}, \xi^-_{|I_r|}) \prod_{s \in \{1,\dots,j\} \setminus X} \xi_j^- \left(\tanh(\phi^{(L,\epsilon)}(h)) - \tanh(\omega(h))\right)\end{aligned} \tag{3.58}$$

for any $j = 1, \dots, k$ with $k = 2, 3, \dots$, and any partition $\{I_1, \dots, I_j\}$. In view of (3.57), we get (3.40) with a suitable M_k for $k \geqslant 2$. ∎

Let us now show that the probability $P_{L,h}(\mathscr{C}^c)$ appearing on the right-hand side of (3.22) and (3.31) converges—within the interval $J_L(\vartheta)$—exponentially fast to zero and that the convergence is uniform in h.

Lemma 3.8. Let $\beta > \beta_c$, $\vartheta > 0$, $0 < \epsilon < \epsilon_0(\vartheta)$. There exist finite constants $\lambda = \lambda(\beta, \vartheta, \epsilon) > 0$ and $L_3 = L_3(\beta, \vartheta, \epsilon)$ such that

$$P_{L,h}(\mathscr{C}^c) \leqslant e^{-\beta\lambda L} \tag{3.59}$$

whenever $L > L_3$ and $h \in J_L(\vartheta)$.

Proof. Let $\beta > \beta_c$, $\vartheta > 0$, $0 < \epsilon < \epsilon_0(\vartheta)$, $L \in \mathbb{N}$, and $h \in J_L(\vartheta)$. Moreover, let

$$\tilde{\mathscr{C}}(Lh, \epsilon) := (m_+(Lh) - \epsilon, m_+(Lh) + \epsilon) \cup (-m^* - \epsilon, -m^* + \epsilon); \tag{3.60}$$

it clearly follows that $\tilde{\mathscr{C}} \subset \mathscr{C}$, i.e. $P_{L,h}(\mathscr{C}^c) \leqslant P_{L,h}(\tilde{\mathscr{C}}^c)$.

With the help of Lemma A.6 one has

$$\inf_{m \in (\tilde{\mathscr{C}}(B,\epsilon))^c} \mathscr{W}_B(m) \geqslant \min\{\mathscr{W}_B(-m^* + \epsilon), \mathscr{W}_B(m(B) - \epsilon), \mathscr{W}_B(m(B) + \epsilon)\} \tag{3.61}$$

for any $B \in \mathbb{R}$. Since $\mathscr{W}_B(m)$ as well as $m(B)$ is continuous in B, the infimum over $\overline{J_L(\vartheta)} = [B_0 - \vartheta, B_0 + \vartheta]$ is attained,

$$\inf_{B \in [B_0 - \vartheta, B_0 + \vartheta]} \mathscr{W}_B(\xi(B)) = \mathscr{W}_{B_\xi}(\xi(B_\xi)) \tag{3.62}$$

for some point $B_\xi \in [B_0 - \vartheta, B_0 + \vartheta]$, where $\xi(B)$ stands for $-m^* + \epsilon$, $m(B) - \epsilon$, or $m(B) + \epsilon$ (the value B_ξ may differ for each of these three functions). Thus, there exist values $B_i \in [B_0 - \vartheta, B_0 + \vartheta]$, $i = 1, 2, 3$, depending on β, ϑ, and ϵ such that

$$\begin{aligned}
&\inf_{m \in (\tilde{\mathscr{C}}(Lh,\epsilon))^c} \mathscr{W}_{Lh}(m) \\
&\quad \geqslant \inf_{h \in \overline{J_L(\vartheta)}} \inf_{m \in (\tilde{\mathscr{C}}(Lh,\epsilon))^c} \mathscr{W}_{Lh}(m) = \inf_{B \in [B_0 - \vartheta, B_0 + \vartheta]} \inf_{m \in (\tilde{\mathscr{C}}(B,\epsilon))^c} \mathscr{W}_B(m) \\
&\quad \geqslant \min\{\mathscr{W}_{B_1}(-m^* + \epsilon), \mathscr{W}_{B_2}(m(B_2) - \epsilon), \mathscr{W}_{B_3}(m(B_3) + \epsilon)\} > 0,
\end{aligned} \tag{3.63}$$

where we used Theorem 3.2 (3) in the last step. Denoting the last minimum by 2λ, Lemma A.4 implies

$$P_{L,h}(\tilde{\mathscr{C}}^c) \leqslant e^{-\beta\lambda L} \tag{3.64}$$

for all L large (depending on β, ϑ, and ϵ). ▮

Using the preceding four lemmas, we shall now prove Theorem 3.3.

Proof of Theorem 3.3. Let $\beta > \beta_c$, $\vartheta > 0$, $h \in J_L(\vartheta)$, and $0 < \epsilon < \epsilon_0(\vartheta)$. We may collect the relations (3.22), (3.31), (3.40), and (3.59) to conclude that there exists $L_4 = L_4(\beta, \vartheta, \epsilon)$ such that for $L > L_4$ the following holds: there exist finite positive constants $\lambda(\beta, \vartheta, \epsilon)$ and D_k, $k \in \mathbb{N}$, such that

$$\left| \frac{1}{(\beta\,|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L \rangle_{L,h} - \mathscr{F}_k(\{T(\omega(h); \xi_j^+, \xi_j^-)\}) \right| \leqslant D_k\, E_L(\epsilon) \tag{3.65}$$

for all $k \in \mathbb{N}$, where

$$E_L(\epsilon) := e^{-\beta\lambda L} + |\Delta m(L h_0(L, \epsilon) + L^{-1/2}) - \Delta| + (1 + 1/\Delta)\,\epsilon > 0. \tag{3.66}$$

Next, we shall use (3.65) to show that the susceptibility $\chi_L(h, \beta)$ attains its maximum over $J_L(\vartheta)$ at a unique point $h_\chi(L)$ for any L sufficiently large (depending on β, ϑ, and ϵ), and thus evaluate its position.

First, notice that

$$\mathscr{F}_k(\{T(x; \xi_j^+, \xi_j^-)\}) = (\Delta m(Lh))^k \frac{d^{k-1} \tanh x}{dx^{k-1}} \tag{3.67}$$

for $k = 2, 3, 4$. Now, let us show that if the point $h_\chi(L)$ exists, then, necessarily, one has $|h_\chi(L) - h_0(L, \epsilon)| < \alpha/|\Lambda_L|$, where $\alpha > 0$ will be specified later.[12] Namely, let us show that, for L sufficiently large (depending on β, ϑ, ϵ, and α),

$$\chi_L(h_0(L, \epsilon), \beta) > \chi_L(h, \beta) \quad \text{once } h \in J_L(\vartheta) \text{ such that } |h - h_0(L, \epsilon)| \geqslant \alpha/|\Lambda_L|. \tag{3.68}$$

This is clear if $|h - h_0(L, \epsilon)| \geqslant L^{-3/2}$: then $|\omega(h)| \geqslant \beta \Delta L^{1/2}$, and (3.65) with $k = 2$ yields

$$\begin{aligned}
&\chi_L(h_0(L, \epsilon), \beta) - \chi_L(h, \beta) \\
&\quad \geqslant (\Delta m(L h_0(L, \epsilon)))^2 - (\Delta m(Lh))^2 \cosh^{-2}(\omega(h)) - 2 D_2 E_L(\epsilon) \\
&\quad \geqslant (2\epsilon_0(\vartheta))^2 - \cosh^{-2}(\beta \Delta L^{1/2}) - 2 D_2 E_L(\epsilon) > 0
\end{aligned} \tag{3.69}$$

[12] We assume that L is large enough (depending on β, ϑ, and α) to ensure that the interval $|h - h_0(L)| \geqslant \alpha/|\Lambda_L|$ fits into $J_L(\vartheta)$.

once $\epsilon > 0$ is small enough (depending on β) and L is large (depending on β, ϑ, and ϵ). On the other hand, if $\alpha/|\Lambda_L| \leqslant |h - h_0(L, \epsilon)| \leqslant L^{-3/2}$, then we have $|\omega(h)| \geqslant \beta\alpha\Delta$, and

$$\begin{aligned}
\chi_L(h_0(L, \epsilon), \beta) - \chi_L(h, \beta) \\
&\geqslant (\Delta m(Lh_0(L, \epsilon)))^2 - (\Delta m(L(h_0(L, \epsilon) + L^{-3/2})))^2 \cosh^{-2}(\omega(h)) \\
&\quad - 2D_2 E_L(\epsilon) \\
&\geqslant (\Delta m(Lh_0(L, \epsilon)))^2 \left[1 - \left(\frac{\Delta m(Lh_0(L, \epsilon) + L^{-1/2})}{\Delta m(Lh_0(L, \epsilon))} \right)^2 \cosh^{-2}(\beta\alpha\Delta) \right] \\
&\quad - 2D_2 E_L(\epsilon) \\
&\geqslant (2\epsilon_0(\vartheta))^2 \left[1 - \left(\frac{\Delta m(Lh_0(L, \epsilon) + L^{-1/2})}{\Delta m(Lh_0(L, \epsilon))} \right)^2 \cosh^{-2}(\beta\alpha\Delta) \right] - 2D_2 E_L(\epsilon).
\end{aligned} \tag{3.70}$$

Taking L so large that

$$\left(\frac{\Delta m(Lh_0(L, \epsilon) + L^{-1/2})}{\Delta m(Lh_0(L, \epsilon))} \right)^2 < 1 + (\cosh^2(\beta\alpha\Delta) - 1)/2 \tag{3.71}$$

(note that the left-hand side above must always be larger than 1), we obtain

$$\chi_L(h_0(L, \epsilon), \beta) - \chi_L(h, \beta) \geqslant (2\epsilon_0)^2\, (1 - \cosh^{-2}(\beta\alpha\Delta))/2 - 2D_2 E_L(\epsilon) > 0 \tag{3.72}$$

whenever $\epsilon > 0$ is small enough (depending on β and α) and L is large (depending on β, ϑ, ϵ, and α). This and (3.69) verify (3.68).

Next, we shall show that the susceptibility $\chi_L(h, \beta)$ is concave on the interval $[h_0(L, \epsilon) - \frac{\alpha}{|\Lambda_L|}, h_0(L, \epsilon) + \frac{\alpha}{|\Lambda_L|}]$ and that its derivative is positive at $h_0(L, \epsilon) - \frac{\alpha}{|\Lambda_L|}$ and negative at $h_0(L, \epsilon) - \frac{\alpha}{|\Lambda_L|}$. Indeed, let us consider $h \in J_L(\vartheta)$ such that $|h - h_0(L, \epsilon)| \leqslant \alpha/|\Lambda_L|$. In view of (3.65) with $k = 3$, we have

$$\begin{aligned}
\frac{1}{\beta\, |\Lambda_L|} \frac{\partial \chi_L(h, \beta)}{\partial h} \bigg|_{h = h_0(L, \epsilon) + \alpha/|\Lambda_L|} &\leqslant (\Delta m(Lh))^3 \frac{d^2 \tanh x}{dx^2} \bigg|_{x = \beta\alpha\Delta} + D_3 E_L(\epsilon) \\
&\leqslant -(2\epsilon_0)^3 \left| \frac{d^2 \tanh x}{dx^2} \right|_{x = \beta\alpha\Delta} + D_3 E_L(\epsilon) < 0
\end{aligned} \tag{3.73}$$

and

$$\frac{1}{\beta\,|\Lambda_L|}\left.\frac{\partial\chi_L(h,\beta)}{\partial h}\right|_{h=h_0(L,\epsilon)-\alpha/|\Lambda_L|} \geqslant (2\epsilon_0)^3\left|\frac{d^2\tanh x}{dx^2}\right|_{x=\beta\alpha\Delta} - D_3E_L(\epsilon)>0 \tag{3.74}$$

for ϵ small (depending on β and α) and L large (depending on β, ϑ, ϵ, and α). Here we used that $\frac{d^2\tanh x}{dx^2}$ is odd and negative for $x>0$. Observing that $\frac{d^3\tanh x}{dx^3}<0$ once $|x|<2A$ for some $A>0$, we choose $\alpha=\frac{A}{\beta\Delta}$: then $|\omega(h)|\leqslant A$, and, using (3.65) with $k=4$, we get

$$\frac{1}{(\beta\,|\Lambda_L|)^2}\frac{\partial^2}{\partial h^2}\chi_L(h)\leqslant -(2\epsilon_0)^4\left|\frac{d^3\tanh x}{dx^3}\right|_{x=A} - D_4E_L(\epsilon)<0 \tag{3.75}$$

for all $|h-h_0(L,\epsilon)|\leqslant\alpha/|\Lambda_L|=A/(\beta\Delta\,|\Lambda_L|)$ whenever ϵ is sufficiently small (depending on β) and L is sufficiently large (depending on β, ϑ, and ϵ). Combined with the fact that the susceptibility $\chi_L(h,\beta)$ is analytic in h, we thus see that the point $h_\chi(L)$ exists, it is unique, and $|h_\chi(L)-h_0(L,\epsilon)|<A/(\beta\Delta\,|\Lambda_L|)$. Thus,

$$\lim_{L\to\infty} L\,h_\chi(L)=B_0 \tag{3.76}$$

due to (3.38), which verifies the first part of (3.13).

Further, let us prove that

$$|h_0(L,\epsilon)-h_\chi(L)|\leqslant\frac{2D_3}{\beta\theta(\epsilon_0/2)^4}\frac{E_L(\epsilon)}{|\Lambda_L|} \tag{3.77}$$

for L large enough (depending on β, ϑ, and ϵ), where $\theta:=-\frac{d^3\tanh x}{dx^3}|_{x=A}>0$. This is trivial if $h_\chi(L)$ happens to coincide with $h_0(L,\epsilon)$. So, let us assume that $h_\chi(L)\neq h_0(L,\epsilon)$. Then the Lagrange mean-value theorem yields

$$\frac{\partial}{\partial h}\chi_L(h,\beta)|_{h=h_0(L,\epsilon)}=(h_0(L,\epsilon)-h_\chi(L))\,\frac{\partial^2}{\partial h^2}\chi_L(h,\beta)|_{h=\bar h} \tag{3.78}$$

for some $\bar h$ between $h_0(L,\epsilon)$ and $h_\chi(L)$ and L large (depending on β, ϑ, and ϵ). By virtue of (3.65) and the bound $|\omega(\bar h)|\leqslant|\omega(h_\chi(L))|<A$, for L large we have

$$\frac{1}{(\beta\,|\Lambda_L|)^2}\frac{\partial^2}{\partial h^2}\chi_L(h,\beta)|_{h=\bar h}\leqslant(2\epsilon_0)^4\left.\frac{d^3\tanh x}{dx^3}\right|_{x=A}+D_4E_L(\epsilon)\leqslant -(2\epsilon_0)^4\,\theta/2, \tag{3.79}$$

yielding a lower bound on its absolute value. On the other hand, the absolute value of $\frac{\partial}{\partial h}\chi_L(h,\beta)$ at $h_0(L,\epsilon)$ can be bounded from above by $D_3\beta\,|\Lambda_L|E_L(\epsilon)$. Both last bounds are valid for L large enough—depending on β, ϑ, and ϵ. As a result, the bound (3.77) follows with the help of (3.78).

Using (3.77), it readily follows that

$$|\tanh(\omega(h))-\tanh(\tilde{\omega}(h))|\leqslant|\omega(h)-\tilde{\omega}(h)|<\frac{2D_3\Delta}{\theta(\epsilon_0/2)^4}\,E_L(\epsilon) \tag{3.80}$$

with

$$\tilde{\omega}(h):=\beta\Delta(h-h_\chi(L))\,|\Lambda_L|, \tag{3.81}$$

for all L large (depending on β, ϑ, ϵ). This in turn implies that for any $k\in\mathbb{N}$ there is a finite positive constant $\tilde{D}_k$ such that

$$|\mathscr{F}_k(\{T(\omega(h);\,\xi_j^+,\xi_j^-)\})-\mathscr{F}_k(\{T(\tilde{\omega}(h);\,\xi_j^+,\xi_j^-)\})|\leqslant\tilde{D}_k\,E_L(\epsilon) \tag{3.82}$$

for all L large (depending on β, ϑ, ϵ), c.f. (3.58). Since the absolute value of the error term $R_L^{(k)}(h)$, $k=1,2$, can be bounded by the sum of the left-hand side of (3.65) and the left-hand side of (3.82), it follows from (3.66) that

$$0\leqslant\overline{\lim_{L\to\infty}}\sup_{h\in J_L(\vartheta)}|R_L^{(k)}(h)|\leqslant(D_k+\tilde{D}_k)\lim_{L\to\infty}E_L(\epsilon)\leqslant(D_k+\tilde{D}_k)(1+1/\Delta)\,\epsilon \tag{3.83}$$

for any $\epsilon>0$, $0<\epsilon<\epsilon_0(\vartheta)$. As a result, $\overline{\lim}_{L\to\infty}\sup_{h\in J_L(\vartheta)}|R_L^{(k)}(h)|=0$, $k=1,2$, and the second claim of (3.13) is verified. ∎

4. EVALUATION OF ERROR TERMS; PROOF OF THEOREM 2.2

In this section, we use the estimates established in ref. 1, 9, 13, 17 to prove Theorem 2.2. Namely, we shall employ the following statements (we formulate them in the form in which we shall need them here).

Theorem 4.1.[1,17] Let $\beta>\beta_c$ and let $\delta\in(0,1/4)$ be given. There exists $L_5=L_5(\beta,\delta)<\infty$ such that for any $L>L_5$ and for any sequence $\{m_L\}$ such that $m_L\in\operatorname{Ran}S_L$, $m_L\geqslant-m^*+L^{-\delta}$, and $\lim_{L\to\infty}m_L\in[-m^*,m^*)$ exists, one has

$$\log P_{L,0}(m_L)\leqslant-\beta(\mathscr{W}_0(m_L)+s_L(m_L))\,L, \tag{4.1}$$

where

$$s_L(m_L) = \begin{cases} O(L^{-(1/4+\delta)/4}) & \text{if} \quad |m_L + m^* - CL^{-\bar{\delta}}| \leqslant L^{-(1/4+\delta)/2} \\ & \text{for some} \quad C > 0 \text{ and } \bar{\delta} \in (0, \delta], \\ O(L^{-1/2}\log L) & \text{otherwise.} \end{cases} \tag{4.2}$$

The first case in (4.2) is a consequence of Theorem 7.4.3 from ref. 17, while the second case is a special case of Theorem 4.3.1 from ref. 1.

The next theorem is a consequence of Theorem 1.5.1 from ref. 9 (c.f. (1.1.2) in ref. 13) and the bound (1.1.1) of ref. 13.

Theorem 4.2. [9, 13] Let $\beta \neq \beta_c$. There exist constants $L_6 = L_6(\beta) < \infty$ and $c_1 = c_1(\beta) > 0$ such that for $L > L_6$ and any $m_L \leqslant \langle S_L \rangle_{L,0}$ one has $\langle S_L \rangle_{L,0} = -m^* + O(L^{-1})$ and

$$\log P_{L,0}(m_L) \leqslant -c_1 \,[(\langle S_L \rangle_{L,0} - m_L) L]^2. \tag{4.3}$$

In addition, we also need this consequence of Theorem C from ref. 13.

Theorem 4.3. [13] Let $\beta > \beta_c$. There exist constants $L_7 = L_7(\beta) < \infty$ and $c_2 = c_2(\beta) > 0$ such that

$$P_{L,0}(m_L) = \frac{c_2}{L}(1 + o_L(1)) \tag{4.4}$$

for all $L > L_7$ and an arbitrary $m_L = \langle S_L \rangle_{L,0} + o(L^{-1}) \in \operatorname{Ran} S_L$ such that $m_L > \langle S_L \rangle_{L,0}$.

Finally, we shall make use of the following lemma.

Lemma 4.4. Let $\beta > \beta_c$ and $\vartheta > 0$. Introducing

$$\eta(\beta, \vartheta) := [m^* - m_+(B_0 + \vartheta)]/2 = \kappa (B_0 + \vartheta)^{-2}/8, \tag{4.5}$$

there exist constants $L_8 = L_8(\beta, \vartheta) < \infty$ and $c_3 = c_3(\beta, \vartheta) > 0$ such that

$$\log P_{L,h}([m^* - \eta, 1]) \leqslant -c_3 \,\beta\, L \tag{4.6}$$

for any $h \in J_L(\vartheta)$ and all $L > L_8$.

Proof. With the help of (3.6) and (3.5), one may easily observe (c.f. the proof of Lemma A.6) that the first derivative of $\mathscr{W}_0$ is strictly decreasing on $(-m^*, m_t)$, whereas it is strictly increasing on (m_t, m^*). This implies that $\mathscr{W}_{Lh}$ is strictly increasing on $(-m^*, m^*)$ once $Lh \leqslant \tau/m^*$:

$$\frac{d\mathscr{W}_{Lh}(m)}{dm} \geqslant \frac{d\mathscr{W}_0(m)}{dm} - \beta\tau/m^* > \frac{d\mathscr{W}_0(m_t)}{dm} - \beta\tau/m^* = 0 \tag{4.7}$$

for all $m \in (-m^*, m^*)$. On the other hand, if $Lh > \tau/m^*$, then $\mathscr{W}_{Lh}$ is strictly increasing on $(m(Lh), m^*)$. Indeed, for $Lh > \tau/m^*$, we have $m(Lh) > m_t$. Hence,

$$\frac{d\mathscr{W}_{Lh}(m)}{dm} = \frac{d\mathscr{W}_0(m)}{dm} - \beta Lh > \frac{d\mathscr{W}_0(m(Lh))}{dm} - \beta Lh = 0 \tag{4.8}$$

for all $m \in (m(Lh), m^*)$. Since $m^* - \eta \in (-m^*, m^*)$ and $m^* - \eta > m(Lh)$ as soon as $Lh \leqslant B_0 + \vartheta$, it follows that

$$\begin{aligned} \inf_{m \in [m^* - \eta, 1]} \mathscr{W}_{Lh}(m) = \mathscr{W}_{Lh}(m^* - \eta) &\geqslant \inf_{h \in \overline{J_L(\vartheta)}} \mathscr{W}_{Lh}(m^* - \eta) \\ &= \inf_{B \in [B_0 - \vartheta, B_0 + \vartheta]} \mathscr{W}_B(m^* - \eta) \end{aligned} \tag{4.9}$$

for any $L \in \mathbb{N}$ and $h \in J_L(\vartheta)$. As $\mathscr{W}_B$ is continuous in B by (3.6) and Proposition 2.1, the infimum on the right-hand side of (4.9) is attained, i.e. it equals $\mathscr{W}_{\tilde{B}}(m^* - \eta)$ for some $\tilde{B} \in [B_0 - \vartheta, B_0 + \vartheta]$. Using $c_3(\beta, \vartheta)$ to denote $\mathscr{W}_{\tilde{B}}(m^* - \eta)/2$, Lemma 1.4 implies the proposition. ▮

We are now ready to prove Theorem 2.2.

4.1. Proof of Theorem 2.2

The proof goes along the same lines as that of Theorem 3.3. However, instead of the large-deviation principle for the sequence $\{P_{L,0}\}$, here we shall take into account a more accurate information on the asymptotic behaviour of the distribution $P_{L,0}$ given above. This will enable us to get explicit rates at which the error terms $R_L^{(0)}$, $R_L^{(1)}(h)$, and $R_L^{(2)}(h)$ tend to zero as $L \to \infty$. For this reason, the parameter $\epsilon > 0$ appearing in the definition of the sets $\mathscr{C}^+$ and $\mathscr{C}^-$, will now be chosen dependent on L, and we use ϵ_L to denote it. Only later shall we specify this dependence precisely—the choice will minimize the above error terms.

Let $\beta > \beta_c$, $\vartheta > 0$, $\delta \in (0, 1/4)$, and let $\eta(\beta, \vartheta) > 0$ be defined by (4.5). We consider a fixed sequence $\{\epsilon_L\}$, $\epsilon_L > 0$, which may depend on δ and β

but not on ϑ such that $\lim_{L\to\infty}\epsilon_L=0$ and $\epsilon_L\geqslant L^{-\delta}$ for all $L\in\mathbb{N}$. It will actually turn out that an optimal choice for our purposes is $\epsilon_L=L^{-\delta}$. Using ϵ_L in the place of ϵ, for a given L we divide the interval $J_L(\vartheta)$ into a *finite* number of sub-intervals as at the beginning of Subsection 3.1. Consequently, the points m_i, $i\in\mathbb{Z}$, will now depend on β and ϵ_L, while the finite numbers N_1 and N_2 as well as the points $B^{(i)}$, $i=-N_1,\dots,N_2$, will depend on β, ϑ, and ϵ_L. We again have

$$J_L(\vartheta)=\bigcup_{i=-N_1}^{N_2}\mathscr{I}_{L,i}^{(\epsilon_L)}. \tag{4.10}$$

Furthermore, for any $h\in J_L(\vartheta)$ we set $\mathscr{C}^+(Lh,\epsilon_L):=(m_i-\epsilon_L, m_{i+1}+\epsilon_L)$ if $h\in\mathscr{I}_{L,i}^{(\epsilon_L)}$, while $\mathscr{C}^-(\epsilon_L):=(-m^*-\epsilon_L/4, -m^*+\epsilon_L/4)$. As before, we shall write $\mathscr{C}(Lh,\epsilon_L)$ for the union $\mathscr{C}^\pm(Lh,\epsilon_L)\cup\mathscr{C}^-(\epsilon_L)$.

In order to verify Theorem 2.2, we first prove three auxiliary lemmas. The first is just an expression of the distribution $P_{L,h}$, $h\in\mathbb{R}$, in terms of $P_{L,0}$.

Lemma 4.5. Let $\beta>0$, $h\in\mathbb{R}$, $L\in\mathbb{N}$, and let us take any $A\in\mathscr{B}(\mathbb{R})$ (the set A may depend on β, h, and L). Then

$$P_{L,h}(A)=\frac{\sum_{m\in A\cap\operatorname{Ran}S_L}e^{\beta h|\Lambda_L|m}P_{L,0}(m)}{\sum_{m'\in\operatorname{Ran}S_L}e^{\beta h|\Lambda_L|m'}P_{L,0}(m')}. \tag{4.11}$$

Proof. Let $\beta>0$, $h\in\mathbb{R}$, $L\in\mathbb{N}$, and $A\in\mathscr{B}(\mathbb{R})$. It suffices to realize that $P_{L,h}(A)=\sum_{m\in A\cap\operatorname{Ran}S_L}P_{L,h}(m)$ and combine the obvious equality

$$P_{L,h}(m)=\frac{e^{\beta h|\Lambda_L|m}}{Z_{L,h}}\sum_{\substack{\sigma_L\in\{-1,1\}^{\Lambda_L}:\\ S_L(\sigma_L)=m}}e^{-\beta H_{L,0}(\sigma_L)}=e^{\beta h|\Lambda_L|m}P_{L,0}(m)\frac{Z_{L,0}}{Z_{L,h}} \tag{4.12}$$

with the fact that $\sum_{m'\in\operatorname{Ran}S_L}P_{L,h}(m')=1$. ∎

The behaviour of the function $\phi^{(L,\epsilon_L)}(h)$ is inspected in the next lemma.

Lemma 4.6. Let $\beta>\beta_c$, $\vartheta>0$, and $\delta\in(0,1/4)$. There exists a finite constant $L_9=L_9(\beta,\vartheta,\delta)$ such that for $L>L_9$ the function $\phi^{(L,\epsilon_L)}(h)$ equals zero within $J_L(\vartheta)$ at a unique point[13] $h_0(L)$. Moreover,

$$|Lh_0(L)-B_0|\leqslant 2(B_0)^3\,\epsilon_L/\kappa. \tag{4.13}$$

[13] We are not denoting the dependence of $h_0(L)$ on ϵ_L. Since we assume that the sequence $\{\epsilon_L\}$ is fixed, this dependence is actually a dependence on L.

Proof. Let $\beta > \beta_c$, $\vartheta > 0$, $\delta \in (0, 1/4)$, and $i = -N_1, \ldots, -1$. With the help of Lemma 4.5 and Theorem 4.1 we may bound

$$\frac{P_{L, B^{(i+1)}/L}(\mathscr{C}^+(B^{(i+1)}, \epsilon_L))}{P_{L, B^{(i+1)}/L}(\mathscr{C}^-(\epsilon_L))} = \frac{\sum_{m \in \mathscr{C}^+(B^{(i+1)}, \epsilon_L) \cap \operatorname{Ran} S_L} e^{\beta B^{(i+1)} L m} P_{L,0}(m)}{\sum_{\hat{m} \in \mathscr{C}^-(\epsilon_L) \cap \operatorname{Ran} S_L} e^{\beta B^{(i+1)} L \hat{m}} P_{L,0}(\hat{m})}$$
$$\leqslant \frac{8L^2 \epsilon_L \max_{m \in \mathscr{C}^+(B^{(i+1)}, \epsilon_L) \cap \operatorname{Ran} S_L} \{e^{\beta B^{(i+1)} L m} P_{L,0}(m)\}}{e^{\beta B^{(i+1)} L \hat{m}_L} P_{L,0}(\hat{m}_L)}$$
$$\leqslant \frac{8L^2 \epsilon_L}{P_{L,0}(\hat{m}_L)} \max_{m \in \mathscr{C}^+(B^{(i+1)}, \epsilon_L)} \{e^{\beta B^{(i+1)} L m - \beta[\mathscr{W}_0(m) + O(L^{-1/2} \log L)] L}\} e^{-\beta B^{(i+1)} L \hat{m}_L} \tag{4.14}$$

for all $L > L_5$ and arbitrary $\hat{m}_L \in \mathscr{C}^-(\epsilon_L) \cap \operatorname{Ran} S_L$. Choosing $\hat{m}_L > \langle S_L \rangle_{L,0}$ such that $\hat{m}_L = \langle S_L \rangle_{L,0} + o(L^{-1}) = -m^* + O(L^{-1})$, Theorem 4.3 and (3.6) yield

$$\frac{P_{L, B^{(i+1)}/L}(\mathscr{C}^+(B^{(i+1)}, \epsilon_L))}{P_{L, B^{(i+1)}/L}(\mathscr{C}^-(\epsilon_L))}$$
$$\leqslant \frac{16L^3 \epsilon_L}{c_2} \max_{m \in (m_i - \epsilon_L, m_{i+1} + \epsilon_L)} e^{-\beta L[\mathscr{W}_0(m) - B^{(i+1)} m - B^{(i+1)}(-m^*) + O(L^{-1/2} \log L)]}$$
$$\leqslant e^{O(\log L)} \max_{m \in [m_i - \epsilon_L, m_{i+1} + \epsilon_L]} e^{-\beta L[\mathscr{W}_{B^{(i+1)}}(m) - \mathscr{W}_{B^{(i+1)}}(-m^*) + O(L^{-1/2} \log L)]} \tag{4.15}$$

once L is large enough (depending on β and δ); we also used the fact that $\mathscr{W}_B(-m^*) = 0$ if $B \leqslant B_0$ by Theorem 3.2 (3). Since $B^{(i+1)} < B_0$, we have

$$\phi^{(L, \epsilon_L)}(B^{(i+1)}/L) \leqslant -\beta L[\min_{m \in [m_i - \epsilon_L, m_{i+1} + \epsilon_L]} \mathscr{W}_{B^{(i+1)}}(m) + O(L^{-1/2} \log L)]/2$$
$$= -\beta L\, [\mathscr{W}_{B^{(i+1)}}(\tilde{m}_i) + O(L^{-1/2} \log L)]/2 \tag{4.16}$$

for all L large (depending on β and δ) and some $\tilde{m}_i \in [m_i - \epsilon_L, m_{i+1} + \epsilon_L]$. When $B^{(i+1)} \leqslant \tau/m^*$, where $\tau/m^* < B_0$, then there is a constant $\zeta(\beta, \vartheta, i) > 0$ such that $\mathscr{W}_{B^{(i+1)}}(x_i) \geqslant \zeta$; this follows from Theorem 3.2 (3) and the bound (3.43). Thus, $\phi^{(L, \epsilon_L)}(B^{(i+1)}/L) \leqslant -\beta \zeta L/4 < 0$ for any L large (depending on β, ϑ, δ, and i). On the other hand, if $B^{(i+1)} > \tau/m^*$, then $\tilde{m}_i = m(B^{(i+1)})$, for the function $\mathscr{W}_B(x)$ has a local minimum at $m(B)$ once $B > \tau/m^*$. Taking into account (3.5) and (3.9), one finds

$$\phi^{(L, \epsilon_L)}(B^{(i+1)}/L) \leqslant -\frac{\beta L}{2} [\mathscr{W}_{B^{(i+1)}}(m(B^{(i+1)})) + O(L^{-1/2} \log L)]$$
$$= -\frac{\beta L}{2} \left[4\tau - \frac{\kappa}{4B^{(i+1)}} - 2m^* B^{(i+1)} + O(L^{-1/2} \log L) \right] \tag{4.17}$$

for all L large enough. Introducing $g(B) := 4\tau - \kappa/(4B) - 2m^*B$ and recalling that $w > 4\tau/3$, one has

$$\frac{dg(B)}{dB} = -(m^* + m(B)) \leqslant -m^*w^2/(8\tau^2) < -2m^*/9 < 0 \qquad (4.18)$$

whenever $B \geqslant \tau/m^*$. Observing that $g(B_0) = 0$, we thus get

$$\begin{aligned}\phi^{(L,\epsilon_L)}(B^{(i+1)}/L) &\leqslant -\beta L\,[g(B^{(0)}) + O(L^{-1/2}\log L)]/2 \\ &= \beta L[(B^{(0)} - B_0)(m^* + m(\tilde{B})) + O(L^{-1/2}\log L)]/2 \qquad (4.19)\end{aligned}$$

for L large and some $\tilde{B} \in (B^{(0)}, B_0)$. From the construction of the interval $\mathscr{I}_{L,-1}^{(\epsilon_L)}$ it is clear that $B_0 - B^{(0)} \geqslant a\epsilon_L$ for some $a(\beta) > 0$ and L large (depending on β and δ). More acurately, the Taylor expansion of $m(B^{(0)})$ around B_0 and the fact that $m(B^{(0)}) = m(B_0) - \epsilon_L/2$ yield $B_0 - B^{(0)} = (B_0)^3\epsilon_L/\kappa + O((\epsilon_L)^2)$. Therefore, we finally obtain

$$\begin{aligned}\phi^{(L,\epsilon_L)}(B^{(i+1)}/L) \leqslant -\beta L\,[&(B_0)^3\,(m^* + m(\tilde{B}))\,L^{-\delta}/(2\kappa) \\ &+ O(L^{-1/2}\log L)]/2 < 0 \qquad (4.20)\end{aligned}$$

for all L large enough (depending on β, ϑ, and δ). As a result, we see that $\phi^{(L,\epsilon_L)}(h) < 0$ for all $h \in \mathscr{I}_{L,i}^{(\epsilon_L)}$, $i = -N_1, \ldots, -1$, once L is sufficiently large (depending on β, ϑ, and δ) because $\phi^{(L,\epsilon_L)}$ is increasing on each $\mathscr{I}_{L,i}^{(\epsilon_L)}$ according to Lemma 3.7 (a). Notice that one may use the above arguments to show that $\phi^{(L,\epsilon_L)}(h) < 0$ for all $h \in \mathscr{I}_{L,0}^{(\epsilon_L)}$ such that $h \leqslant (B_0 + B^{(0)})/(2L)$ whenever L is large (depending on β, ϑ, and δ).

Next, let us consider $i = 1, \ldots, N_2$. Taking $\hat{m}_L \in \mathscr{C}^+(B^{(i)}, \epsilon_L) \cap \operatorname{Ran} S_L$ such that $\hat{m}_L = m(B^{(i)}) + O(L^{-2})$, by virtue of Theorem 4.1, Lemma 4.5, and the relations (3.6) and (3.9), we bound

$$\begin{aligned}\frac{P_{L,B^{(i)}/L}(\mathscr{C}^-(\epsilon_L))}{P_{L,B^{(i)}/L}(\mathscr{C}^+(B^{(i)},\epsilon_L))} &\leqslant \frac{e^{\beta B^{(i)}L(-m^* + \epsilon_L/4)}\,P_{L,0}(\mathscr{C}^-(\epsilon_L))}{e^{\beta B^{(i)}L\hat{m}_L}\,P_{L,0}(\hat{m}_L)} \\ &\leqslant e^{\beta L\,[\mathscr{W}_{B^{(i)}}(m(B^{(i)})) - \mathscr{W}_{B^{(i)}}(-m^*) + B^{(i)}\epsilon_L/4 + O(L^{-1/2}\log L)]} \\ &= e^{-\beta L\,[\mathscr{W}_{B^{(i)}}(-m^*) - B^{(i)}\epsilon_L/4 + O(L^{-1/2}\log L)]} = e^{-\beta L\,[-g(B^{(i)}) - B^{(i)}\epsilon_L/4 + O(L^{-1/2}\log L)]}\end{aligned} \qquad (4.21)$$

for all $L > L_5$. Since

$$\frac{d}{dB}(-g(B) - B\epsilon_L/4) = m^* + m(B) - \epsilon_L/4 \geqslant 2m^*/9 - \epsilon_L/4 \geqslant m^*/9 > 0 \qquad (4.22)$$

for all L large (depending on β and δ), it follows that

$$\frac{P_{L, B^{(i)}/L}(\mathscr{C}^{-}(\epsilon_L))}{P_{L, B^{(i)}/L}(\mathscr{C}^{\pm}(B^{(i)}, \epsilon_L))} \leqslant e^{-\beta L\,[-g(B^{(1)})-B^{(1)}\epsilon_L/4+O(L^{-1/2}\log L)]} \tag{4.23}$$

once L is large enough (depending on β and δ). As $m(B^{(1)}) = m(B_0) + \epsilon_L/2$, we have $B^{(1)} - B_0 = O(\epsilon_L)$. Namely, the Taylor expansion of $m(B^{(1)})$ around B_0 yields $B^{(1)} - B_0 = (B_0)^3 \epsilon_L/\kappa + O((\epsilon_L)^2)$. Hence,

$$\begin{aligned} -g(B^{(1)}) - B^{(1)}\epsilon_L/4 &= (B^{(1)} - B_0)(m^* + m(B_0)) - B_0\epsilon_L/4 + O((\epsilon_L)^2) \\ &= (B_0)^3\,(2m^* - \kappa/(2B_0)^2)\,\epsilon_L/\kappa - B_0\epsilon_L/4 + O((\epsilon_L)^2) \end{aligned} \tag{4.24}$$

by the Taylor expansion of $g(B^{(1)})$ around B_0. Recalling that $\kappa < 4m^*(B_0)^2$, the inequality $(2m^* - \kappa/(2B_0)^2)\,\kappa > 1/(2B_0)^2$ is true. Thus,

$$\begin{aligned} -g(B^{(1)}) - B^{(1)}\epsilon_L/4 &= B_0[(B_0)^2\,(2m^* - \kappa/(2B_0)^2)/\kappa - 1/4]\,\epsilon_L + O((\epsilon_L)^2) \\ &> B_0[(B_0)^2\,(2m^* - \kappa/(2B_0)^2)/\kappa - 1/4]\,\epsilon_L/2 \end{aligned} \tag{4.25}$$

for all L large (depending on β and δ). Combined with (4.23), we obtain that $\phi^{(L,\epsilon_L)}(h) > 0$ for all $h \in \mathscr{I}_{L,i}^{(\epsilon_L)}$, $i = 1, \ldots, N_2$, once L is sufficiently large (depending on β, ϑ, and δ).

In addition, the above may also be employed to show that $\phi^{(L,\epsilon_L)}(h) > 0$ for any $h \in \mathscr{I}_{L,0}^{(\epsilon_L)}$ such that $h \geqslant (B^{(1)} - (\epsilon_L)^2)/L$, say, and any L large (depending on β, ϑ, and δ). Recalling that, for L large, we have $\phi^{(L,\epsilon_L)}(h) < 0$ for all $h \in \mathscr{I}_{L,0}^{(\epsilon_L)}$ such that $h \leqslant (B_0 + B^{(0)})/(2L)$, the fact that $\phi^{(L,\epsilon_L)}$ is analytic and increasing on $\mathscr{I}_{L,0}^{(\epsilon_L)}$, see Lemma 3.7 (a), implies that the point $h_0(L) \in \mathscr{I}_{L,0}^{(\epsilon_L)}$ exists, it is unique, and

$$B_0/L - h_0(L) < (B_0 - B^{(0)})/(2L) \leqslant (B_0)^3\,\epsilon_L/(\kappa L) \tag{4.26}$$

and

$$h_0(L) - B_0/L < (B^{(1)} - B_0)/L \leqslant 2(B_0)^3\,\epsilon_L/(\kappa L) \tag{4.27}$$

once L is large enough (depending on β, ϑ, and δ). ∎

Finally, in the following lemma, we establish a uniform bound on the probability $P_{L,h}(\mathscr{C}^c)$ for all $h \in J_L(\vartheta)$ analogous to (3.59).

Lemma 4.7. Let $\beta > \beta_c$, $\vartheta > 0$, $\delta \in (0, 1/4)$, and $h \in J_L(\vartheta)$. There exists $L_{10} = L_{10}(\beta, \vartheta, \delta) < \infty$ such that

$$P_{L,h}(\mathscr{C}^c) \leqslant 3e^{-\beta(\tau/m^*)^3 L(\epsilon_L)^2/\kappa} \tag{4.28}$$

as long as $L > L_{12}$ and $h \in J_L(\vartheta)$.

Proof. Let $\beta > \beta_c$, $\vartheta > 0$, $\delta \in (0, 1/4)$, and $h \in J_L(\vartheta)$. Clearly, we have $m^* - \eta > \sup \mathscr{C}^+(Lh, \epsilon_L)$ once L is large (depending on β and ϑ), where $\eta(\beta, \vartheta)$ is given by (4.5). Then, obviously,

$$P_{L,h}(\mathscr{C}^c) \leqslant P_{L,h}([-1, -m^* - \epsilon_L/4]) + P_{L,h}(A_{Lh}(\epsilon_L)) + P_{L,h}([a_2, 1]) \tag{4.29}$$

with

$$A_{Lh}(\epsilon_L) := [a_1, m_+(Lh) - \epsilon_L] \cup [m_+(Lh) + \epsilon_L, a_2], \tag{4.30}$$

where we used the shorthands $a_1 := -m^* + \epsilon_L/4$ and $a_2 := m^* - \eta$. Next, we shall uniformly bound each of the three above probabilities separately.

Let $\tilde{m}_L \in \operatorname{Ran} S_L$ be such that $\tilde{m}_L > \langle S_L \rangle_{L,0}$ and $\tilde{m}_L = \langle S_L \rangle_{L,0} + O(L^{-2})$. Restricting the sum in the denominator of (4.11) to a single term with $m = \tilde{m}_L$ and using Theorem 4.2, Theorem 4.3, and Lemma 4.5, we get

$$\begin{aligned} &P_{L,h}([-1, -m^* - \epsilon_L/4]) \\ &\quad \leqslant \frac{L^2 \max_{m \in [-1, -m^* - \epsilon_L/4] \cap \operatorname{Ran} S_L} \{ e^{\beta h |\Lambda_L| m - c_1 [(\langle S_L \rangle_{L,0} - m) L]^2} \}}{e^{\beta h |\Lambda_L| \tilde{m}_L} P_{L,0}(\tilde{m}_L)} \\ &\quad \leqslant \frac{2L^3}{c_2} \max_{m \in [-1, -m^* - \epsilon_L/4]} \{ e^{\beta h |\Lambda_L| m - c_1 [(\langle S_L \rangle_{L,0} - m) L]^2} \} \, e^{-\beta h |\Lambda_L| (\langle S_L \rangle_{L,0} + O(L^{-2}))} \end{aligned} \tag{4.31}$$

for all $L > \max\{L_6, L_7\}$. The function $q(m) := \beta h |\Lambda_L| m - c_1 [(\langle S_L \rangle_{L,0} - m) L]^2$ is increasing on $(-\infty, -m^* - \epsilon_L/4]$ if L is large enough (depending on β and ϑ): then

$$\begin{aligned} \frac{dq(m)}{dm} &= \beta h |\Lambda_L| + 2c_1 (\langle S_L \rangle_{L,0} - m) L^2 \geqslant \beta(B_0 - \vartheta) L + 2c_1(\epsilon_L/4 + O(L^{-1})) L^2 \\ &= [\beta(B_0 - \vartheta) + c_1 \epsilon_L L/2 + O(1)] L > 0 \end{aligned} \tag{4.32}$$

because $\langle S_L\rangle_{L,0} = -m^* + O(L^{-1})$ and $\epsilon_L L > L^{3/4}$. As a consequence,

$$P_{L,h}([-1, -m^* - \epsilon_L/4]) \leqslant \frac{2L^3}{c_2} e^{q(-m^* - \epsilon_l/4) - \beta h |\Lambda_L| (\langle S_L\rangle_{L,0} + O(L^{-2}))}$$
$$= e^{O(\log L)} e^{\beta h |\Lambda_L| (\epsilon_l/4 + O(L^{-1})) - c_1(\epsilon_L/4 + O(L^{-1}))^2 L^2}$$
$$\leqslant e^{O(\log L) + \beta(B_0 + \vartheta) L\epsilon_L/2 - c_1(L\epsilon_L/6)^2} \leqslant e^{-c_1(L\epsilon_L/8)^2} \leqslant e^{-L^{3/2}}. \tag{4.33}$$

for all L large enough (depending on β and ϑ).

Next, Theorem 4.1 and Lemma 4.5 yield

$$P_{L,h}(A_{Lh}(\epsilon_L)) \leqslant 2L^2 \frac{\max_{m \in A_{Lh}(\epsilon_L) \cap \operatorname{Ran} S_L} e^{\beta h |\Lambda_L| m} P_{L,0}(m)}{e^{\beta h |\Lambda_L| m_L} P_{L,0}(m_L)}$$
$$\leqslant 2L^2 \frac{\max_{m \in A_{Lh}(\epsilon_L) \cap \operatorname{Ran} S_L} e^{-\beta L[\mathscr{W}_0(m) - Lhm + s_L(m)]}}{e^{\beta h |\Lambda_L| m_L} P_{L,0}(m_L)} \tag{4.34}$$

for any $m_L \in \operatorname{Ran} S_L$ and L large (depending on β, ϑ, and δ). Introducing $f_{Lh}(m) := \mathscr{W}_0(m) - Lhm$ and denoting $a^* := -m^* + \epsilon_0(\vartheta)/2$, we may now write

$$P_{L,h}(A_{Lh}(\epsilon_L)) \leqslant \frac{2L^2}{e^{\beta h |\Lambda_L| m_L} P_{L,0}(m_L)} \max\Big\{ \max_{m \in [a_1, a^*]} e^{-\beta L[f_{Lh}(m) + O(L^{-(1/4+\delta)/4})]},$$
$$\max_{m \in [a^*, a_2] \cap A_{Lh}(\epsilon_L)} e^{-\beta L[f_{Lh}(m) + O(L^{-1/2} \log L)]}\Big\} \tag{4.35}$$

if L is large (depending on β, ϑ, and δ). Using Lemma A.6, for L large (depending on β, ϑ, and δ) we obtain

$$P_{L,h}(A_{Lh}(\epsilon_L)) \leqslant \frac{2L^2}{e^{\beta h |\Lambda_L| m_L} P_{L,0}(m_L)} e^{-\beta L[f_{Lh}(a_1) + O(L^{-(1/4+\delta)/4})]} \tag{4.36}$$

if $Lh \leqslant \tau/m^*$, whereas

$$P_{L,h}(A_{Lh}(\epsilon_L)) \leqslant \frac{2L^2}{e^{\beta h |\Lambda_L| m_L} P_{L,0}(m_L)} \max\Big\{ \max_{m \in [a_1, a^*]} e^{-\beta L[f_{Lh}(m) + O(L^{-(1/4+\delta)/4})]},$$
$$\max_{m \in [a^*, a_2] \cap A_{Lh}(\epsilon_L)} e^{-\beta L[f_{Lh}(m) + O(L^{-1/2} \log L)]}\Big\}$$
$$= \frac{2L^2}{e^{\beta h |\Lambda_L| m_L} P_{L,0}(m_L)} \max\Big\{ e^{-\beta L[f_{Lh}(a_1) + O(L^{-(1/4+\delta)/4})]},$$
$$e^{-\beta L[f_{Lh}(m(Lh) \pm \epsilon_L) + O(L^{-1/2} \log L)]}\Big\} \tag{4.37}$$

whenever $Lh \geqslant \tau/m^*$. In the former case, let us take $m_L > \langle S_L \rangle_{L,0}$ such that $m_L = \langle S_L \rangle_{L,0} + O(L^{-2}) = -m^* + O(L^{-1})$. Then, in view of Theorem 4.1 and (3.5),

$$
\begin{aligned}
P_{L,h}(A_{Lh}(\epsilon_L)) &\leqslant \frac{2L^2 e^{-\beta L[w\sqrt{\epsilon_L/(8m^*)} - Lh(-m^* + \epsilon_L/4) + O(L^{-(1/4+\delta)/4})]}}{c_2\, e^{\beta h |\Lambda_L|(-m^* + O(L^{-1}))}\,(1+o_L(1))/L} \\
&= e^{-\beta L[w\sqrt{\epsilon_L/(8m^*)} - Lh(\epsilon_L/4 + O(L^{-1})) + O(L^{-(1/4+\delta)/4}) + O(L^{-1}\log L)]} \\
&\leqslant e^{-\beta w L\sqrt{\epsilon_L/(8m^*)}\,[1 + O(\sqrt{\epsilon_L}) + O(L^{-(1/4-\delta)/4})]} \leqslant e^{-\beta w L\sqrt{\epsilon_L/(12m^*)}}
\end{aligned} \tag{4.38}
$$

for all L large (depending on β, ϑ, and δ). In the latter case, if the maximum in (4.37) coincides with the first term, we use the same procedure as above to get (4.38). However, if the maximum in (4.37) coincides with the second term, we then take $m_L = m(Lh) + O(L^{-2})$ and, by virtue of Theorem 4.1 and (3.5) we find

$$
\begin{aligned}
P_{L,h}(A_{Lh}(\epsilon_L)) &\leqslant 2L^2\, e^{-\beta L\{[\mathscr{W}_0(m(Lh) \pm \epsilon_L) - \mathscr{W}_0(m(Lh) + O(L^{-2}))] \pm Lh\epsilon_L + O(L^{-1/2}\log L)\}} \\
&= e^{-\beta L\{-\frac{\kappa}{2Lh}[\sqrt{1+(2Lh)^2\epsilon_L/\kappa} - \sqrt{1+O(L^{-2})}] \pm Lh\epsilon_L + O(L^{-1/2}\log L)\}} \\
&\leqslant e^{-\beta L[2(Lh)^3(\epsilon_L)^2/\kappa + O((\epsilon_L)^3) + O(L^{-1/2}\log L)]} \\
&\leqslant e^{-2\beta(Lh)^3 L(\epsilon_L)^2/\kappa[1 + O(\epsilon_L) + O(L^{-1/4}\log L)]} \leqslant e^{-\beta(\tau/m^*)^3 L(\epsilon_L)^2/\kappa}
\end{aligned} \tag{4.39}
$$

once L is large (depending on β, ϑ, and δ). Taking into account Lemma 4.4, we may conclude that

$$
P_{L,h}(\mathscr{C}^c) \leqslant 3\max\{e^{-\beta w L\sqrt{\epsilon_L/(12m^*)}}, e^{-\beta(\tau/m^*)^3 L(\epsilon_L)^2/\kappa}\} = 3e^{-\beta(\tau/m^*)^3 L(\epsilon_L)^2/\kappa} \tag{4.40}
$$

for all sufficiently large L (depending on β, ϑ, and δ). ∎

Now, we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. By virtue of the Lemma 3.4 and Lemma 3.6, there exist finite constants C_k, K_k, $k \in \mathbb{N}$, and $L_{11} = L_{11}(\beta, \vartheta, \delta)$ such that for all $\beta > \beta_c$, $\vartheta > 0$, and $\delta \in (0, 1/4)$ we have

$$
\begin{aligned}
&\left| \frac{1}{(\beta\,|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L \rangle_{L,h} - \mathscr{F}_k(\{T(\phi^{(L,\epsilon_L)}(h); \xi_j^+, \xi_j^-)\}) \right| \\
&\qquad \leqslant C_k\, P_{L,h}(\mathscr{C}^c) + K_k\, \epsilon_L
\end{aligned} \tag{4.41}
$$

whenever $L > L_{11}$ and $h \in J_L(\vartheta)$. Here $T(x; a, b)$ and $\xi_j^\pm$ are given by (3.23) and (3.32), respectively. Moreover, Lemma 4.6 yields

$$\begin{aligned} &|\Delta m(Lh_0(L) + L^{-1/2}) - \Delta| \\ &\quad = |m(Lh_0(L)) - m(B_0)|/2 + O(L^{-1/2}) = \epsilon_L/4 + O(L^{-1/2}) \end{aligned} \tag{4.42}$$

for all $L > L_9$ by the Taylor expansion and the fact that $h_0(L) \in \mathscr{I}_{L,0}^{(\epsilon_L)}$. Combined with Lemma 3.7 (c), we may conclude that there exists finite constants M_k, $k \in \mathbb{N}$, and $L_{12} = L_{12}(\beta, \vartheta, \delta)$ such that for all $\beta > \beta_c$, $\vartheta > 0$, and $\delta \in (0, 1/4)$ one has

$$|\mathscr{F}_k(\{T(\phi^{(L,\epsilon_L)}(h); \xi_j^+, \xi_j^-)\}) - \mathscr{F}_k(\{T(\omega(h); \xi_j^+, \xi_j^-)\})| \leqslant 2M_k \epsilon_L/\Delta \tag{4.43}$$

once $L > L_{12}$ and $h \in J_L(\vartheta)$, where $\omega(h) := \beta\Delta(h - h_0(L))\,|\Lambda_L|$. Taking into account Lemma 4.7, we may therefore conclude that there exist finite constants D_k, $k \in \mathbb{N}$, and $L_{13} = L_{13}(\beta, \vartheta, \delta)$ such that

$$\left| \frac{1}{(\beta\,|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L \rangle_{L,h} - \mathscr{F}_k(\{T(\omega(h); \xi_j^+, \xi_j^-)\}) \right| \leqslant D_k\, E_L(\epsilon_L) \tag{4.44}$$

for all $\beta > \beta_c$, $\vartheta > 0$, and $\delta \in (0, 1/4)$ as long as $L > L_{13}$ and $h \in J_L(\vartheta)$, where

$$E_L(\epsilon_L) := e^{-\beta(\tau/m^*)^3\, L(\epsilon_L)^2/\kappa} + (1 + 1/\Delta)\,\epsilon_L > 0. \tag{4.45}$$

Let $\beta > \beta_c$, $\vartheta > 0$, $\delta \in (0, 1/4)$, and $h \in J_L(\vartheta)$. At this point, we shall choose ϵ_L to minimize $E_L(\epsilon_L)$. Namely, let us take

$$\epsilon_L = L^{-\delta}. \tag{4.46}$$

As a consequence,

$$E_L(\epsilon_L) \leqslant 2(1 + 1/\Delta)\, L^{-\delta} \tag{4.47}$$

for all L sufficiently large (depending on β, ϑ, and δ).

According to Theorem 3.3, there is a unique point $h_\chi(L) \in J_L(\vartheta)$ such that if L is large (depending on β, ϑ, and δ), then the susceptibility $\chi_L(h, \beta)$ attains maximum over $J_L(\vartheta)$ at $h_\chi(L)$. In addition, with the help of (4.44) and (4.47) and using the same arguments that led to (3.77), one may bound

$$|h_0(L) - h_\chi(L)| \leqslant \frac{4D_3(1 + 1/\Delta)}{\beta\theta(\epsilon_0/2)^4} \frac{L^{-\delta}}{|\Lambda_L|}, \tag{4.48}$$

for L large enough (depending on β, ϑ, and δ). Hence,

$$|\tanh(\omega(h))-\tanh(\tilde{\omega}(h))| \leqslant |\omega(h)-\tilde{\omega}(h)| < \frac{4D_3(1+\Delta)}{\theta(\epsilon_0/2)^4} L^{-\delta}, \tag{4.49}$$

where

$$\tilde{\omega}(h) := \beta\Delta(h-h_\chi(L))\,|\Lambda_L|, \tag{4.50}$$

for all L large (depending on β, ϑ, δ). This in turn implies that for any $k \in \mathbb{N}$ there is a finite constant $\tilde{D}_k$ such that, with $\mathscr{F}_k$ defined in (3.30), one has

$$|\mathscr{F}_k(\{T(\omega(h); \xi_j^+, \xi_j^-)\}) - \mathscr{F}_k(\{T(\tilde{\omega}(h); \xi_j^+, \xi_j^-)\})| \leqslant \tilde{D}_k\, L^{-\delta} \tag{4.51}$$

for all L large (depending on β, ϑ, and δ), c.f. (3.58). Combined with (4.44), (4.47) and (3.67), Theorem 2.2 follows. ▮

APPENDIX A. TECHNICAL LEMMAS

Here we collect various technical lemmas, some of them rather standard.

Lemma A.1. Let $\psi_r\colon \mathbb{R} \to \mathbb{R}$, $r = 1, 2$, be two C^∞ functions. Then, for any $k \in \mathbb{N}$,

$$\frac{d^k\psi_2(\psi_1(x))}{dx^k} = \sum_{i=1}^{k} \frac{d^i\psi_2(y)}{dy^i}\bigg|_{y=\psi_1(x)} \sum_{\{I_1,\ldots,I_i\}} \prod_{j=1}^{i} \frac{d^{|I_j|}\psi_1(x)}{dx^{|I_j|}},$$

where the second sum runs over all partitions $\{I_1, \ldots, I_i\}$, $i = 1, \ldots, k$, of the set $\{1, \ldots, k\}$ and $|I_j|$, $j = 1\ldots, i$, is the cardinality of I_j.

Proof. By induction on $k \in \mathbb{N}$. ▮

Lemma A.2. Let $h \in \mathbb{R}$, $L \in \mathbb{N}$, $\beta > 0$. Given any $k = 2, 3\ldots$, there exists a finite positive constant C_k such that, with $\mathscr{F}_k$ defined in (3.30),

$$\left|\frac{1}{(\beta\,|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L\rangle_{L,h} - \mathscr{F}_k(\{\langle (S_L)^i \mid A\rangle_{L,h}\})\right| \leqslant C_k P_{L,h}(A^c) \tag{A.1}$$

for any set $A \in \mathscr{B}(\mathbb{R})$ for which $Z_{L,h}(A) > 0$ (the set A may depend on h).

Proof. Let $h \in \mathbb{R}$, $L \in \mathbb{N}$, $\beta > 0$, and $k \in \mathbb{N}$. Let $A \in \mathscr{B}(\mathbb{R})$ be given (it may depend on h) such that $Z_{L,h}(A) > 0$. Since

$$\frac{\partial^n}{\partial h^n} Z_{L,h} = (\beta\,|\Lambda_L|)^n \,\langle (S_L)^n \rangle_{L,h}\, Z_{L,h} \tag{A.2}$$

for all $n \in \mathbb{N}$, Lemma A.1 applied to $\psi_1(h) = Z_{L,h}$ and $\psi_2(x) = \log x$ readily yields

$$\frac{1}{(\beta\,|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L \rangle_{L,h} = \frac{1}{(\beta\,|\Lambda_L|)^k} \frac{\partial^k}{\partial h^k} \log Z_{L,h} = \mathscr{F}_k(\{\langle (S_L)^i \rangle_{L,h}\}). \tag{A.3}$$

Observing that

$$\begin{aligned}\langle (S_L)^n \rangle_{L,h} &= \langle (S_L)^n \mid A \rangle_{L,h}\, P_{L,h}(A) + \langle (S_L)^n \mid A^c \rangle_{L,h}\, P_{L,h}(A^c)\\ &= \langle (S_L)^n \mid A \rangle_{L,h} + (\langle (S_L)^n \mid A^c \rangle_{L,h} - \langle (S_L)^n \mid A \rangle_{L,h})\, P_{L,h}(A^c)\end{aligned} \tag{A.4}$$

for any $n \in \mathbb{N}$, it follows that

$$\begin{aligned}\prod_{j=1}^{i} \langle (S_L)^{|I_j|} \rangle_{L,h} = \prod_{j=1}^{i} \langle (S_L)^{|I_j|} \mid A \rangle_{L,h} + \sum_{\substack{X \subset \{1,\dots,i\}:\\ X \neq \{1,\dots,i\}}} \prod_{r \in X} \langle (S_L)^{|I_r|} \mid A \rangle_{L,h}\\ \times \prod_{s \in \{1,\dots,i\} \setminus X} (\langle (S_L)^{|I_s|} \mid A^c \rangle_{L,h} - \langle (S_L)^{|I_s|} \mid A \rangle_{L,h})\, P_{L,h}(A^c)\end{aligned} \tag{A.5}$$

for all partitions $\{I_1, \dots, I_i\}$ of $\{1, \dots, k\}$. Realizing that $|\langle S_L \mid A \rangle_{L,h}| \leqslant 1$ as well as $|\langle S_L \mid A^c \rangle_{L,h}| \leqslant 1$ and that $P_{L,h}(A^c) \leqslant 1$, the relations (A.3) and (A.5) imply the lemma. ∎

Lemma A.3 [Lemma 6.1 from ref. 4]. Let $x_1, x_2 \in \mathbb{R}$ be either both positive or both negative. Then

$$|\tanh x_1 - \tanh x_2| \leqslant \min\left\{\frac{\tanh x_1}{x_1}, \frac{\tanh x_2}{x_2}\right\} |x_1 - x_2|. \tag{A.6}$$

Proof. Let $x_1, x_2 \in \mathbb{R}$ be given. Without loss of generality, we may suppose that $x_1 > x_2 > 0$. Then $\tanh x_1 > \tanh x_2$ and $\frac{\tanh x_1}{x_1} < \frac{\tanh x_2}{x_2}$. Thus,

$$|\tanh x_1 - \tanh x_2| \frac{x_1}{\tanh x_1} = \left(1 - \frac{\tanh x_2}{\tanh x_1}\right)|x_1| < \left|1 - \frac{x_2}{x_1}\right| |x_1| = |x_1 - x_2|. \quad ∎$$

Lemma A.4. Let $\beta > \beta_c$, $\vartheta > 0$, and $L \in \mathbb{N}$. Let $F \subset \mathbb{R}$ be a closed set (which may depend on h and L) and let us assume that there exists a constant $c > 0$ independent of L such that

$$\inf_{h \in \overline{J_L(\vartheta)}} \inf_{m \in F} \mathscr{W}_{Lh}(m) \geqslant c \qquad \text{for all} \quad L \in \mathbb{N}. \tag{A.7}$$

Then there exists $L_0(\beta, \vartheta, c)$ such that

$$\sup_{h \in \overline{J_L(\vartheta)}} P_{L,h}(F) \leqslant e^{-\beta c L/2} \qquad \text{once} \quad L > L_0. \tag{A.8}$$

Remark A.5. The claim of Lemma A.4 reminds of a standard result from the theory of large deviations. Namely, in view of Theorem 3.2 (2), the condition

$$\inf_{m \in F} \mathscr{W}_B(m) \geqslant c > 0 \tag{A.9}$$

implies, for any sequence $\{h_L\}$ for which $\lim_{L\to\infty} Lh_L = B \in \mathbb{R}$, the bound

$$P_{L,h_L}(F) \leqslant e^{-\beta c L/2} \tag{A.10}$$

for all L large (depending on β, ϑ, and c). However, in our analysis the bound (A.10) is insufficient. Namely, it is confined to the sequences $\{h_L\}$ of the above type, whereas Lemma A.4 provides a bound for the whole interval $J_L(\vartheta)$.

Proof. Let $\delta > 0$, $h \in J_L(\vartheta)$, and a closed $F \subset \mathbb{R}$ be given.

First, let us show that there exists a finite positive integer $1 \leqslant N(\delta) \leqslant 3/\delta$ such that

$$P_{L,h}(F) \leqslant N(\delta)\, e^{2\beta\delta\vartheta L} \frac{\sup_{m \in F} \{e^{\beta |\Lambda_L| hm} P_{L,0}(\mathscr{U}_\delta(m))\}}{\sup_{m \in \mathbb{R}} \{e^{\beta |\Lambda_L| hm} P_{L,0}(\mathscr{U}_\delta(m))\}} \tag{A.11}$$

with $\mathscr{U}_\delta(m) := (m-\delta, m+\delta)$. In view of (3.14),

$$Z_{L,h} \geqslant \sum_{\sigma_L \in \Omega_L : S_L(\sigma_L) \in \mathscr{U}_\delta(m)} e^{-\beta H_{L,0}(\sigma_L) + \beta |\Lambda_L| h S_L(\sigma_L)} \geqslant e^{\beta |\Lambda_L| (hm - \delta\vartheta/L)} Z_{L,0}(\mathscr{U}_\delta(m))$$

for any $m \in \mathbb{R}$. Hence,

$$Z_{L,h} \geqslant e^{-\beta\delta\vartheta L} \sup_{m \in \mathbb{R}} \{e^{\beta |\Lambda_L| hm} Z_{L,0}(\mathscr{U}_\delta(m))\}. \tag{A.12}$$

On the other hand, one may find $1 \leqslant N(\delta) \leqslant 3/\delta$ and $m_1, ..., m_{N(\delta)} \in [-1, 1]$ such that $[-1, 1] \subset \bigcup_{i=1}^{N(\delta)} \mathscr{U}_\delta(m_i)$. As a consequence,

$$Z_{L,h}(F) = Z_{L,h}(F \cap [-1, 1]) \leqslant \sum_{i=1}^{N(\delta)} \sum_{\substack{\sigma_L \in \Omega_L: \\ S_L(\sigma_L) \in F \cap \mathscr{U}_\delta(m_i)}} e^{-\beta H_{L,0}(\sigma_L) + \beta |\Lambda_L| h S_L(\sigma_L)}$$
$$\leqslant N(\delta) \max_{1 \leqslant i \leqslant N(\delta)} e^{\beta |\Lambda_L| (hm_i + \delta\vartheta/L)} Z_{L,0}(F \cap \mathscr{U}_\delta(m_i)). \tag{A.13}$$

Thus,

$$Z_{L,h}(F) \leqslant N(\delta) e^{\beta\delta\vartheta L} \sup_{m \in F} \{e^{\beta |\Lambda_L| hm} Z_{L,0}(\mathscr{U}_\delta(m))\}. \tag{A.14}$$

Combining (A.12) with (A.14) and (3.15), we arrive at (A.11).

Further, let $m \in \mathbb{R}$ be arbitrary. The function $g: \delta \mapsto \inf_{\mathscr{U}_\delta(m)} \mathscr{W}_0$ is obviously non-increasing. So, considering the closure $\overline{\mathscr{U}_\delta(m)} = [m-\delta, m+\delta]$, the equality $\inf_{\mathscr{U}_\delta(m)} \mathscr{W}_0 = \inf_{\overline{\mathscr{U}_\delta(m)}} \mathscr{W}_0$ holds for all the continuity points of g. Recalling that $(P_{L,0})^{1/L} \to e^{-\beta \mathscr{W}_0}$ with $\mathscr{W}_0$ defined by (3.5), we get that the limit

$$\beta \inf_{\mathscr{U}_\delta(m)} \mathscr{W}_0 = -\lim_{L \to \infty} \frac{1}{L} \log P_{L,0}(\mathscr{U}_\delta(m)) \tag{A.15}$$

exists for almost all $\delta > 0$. Because $\lim_{\delta \to 0^+} \inf_{\mathscr{U}_\delta(m)} \mathscr{W}_0 = \mathscr{W}_0(m)$ due to the lower semi-continuity of $\mathscr{W}_0$, it follows that, given any $\varepsilon > 0$, there is $L_0(\varepsilon)$ finite and $\delta_0(\varepsilon) > 0$ such that

$$-\mathscr{W}_0(m) - \varepsilon < \frac{1}{\beta L} \log P_{L,0}(\mathscr{U}_\delta(m)) < -\mathscr{W}_0(m) + 2\varepsilon \tag{A.16}$$

for almost all $0 < \delta < \delta_0(\varepsilon)$ and $L > L_0(\varepsilon)$. In view of (A.11) and (3.6), we may conclude that, for any $\varepsilon > 0$ and almost all $0 < \delta < \delta_0(\varepsilon)$, the bound

$$\frac{1}{\beta L} \sup_{h \in \overline{J_L(\vartheta)}} \log P_{L,h}(F) \leqslant \sup_{h \in \overline{J_L(\vartheta)}} \sup_{m \in F} \{Lhm - \mathscr{W}_0(m) - \mathscr{W}_0^*(Lh)\}$$
$$+ 2\vartheta\delta + 3\varepsilon + \frac{\log(3/\delta)}{\beta L}$$
$$= -\inf_{h \in \overline{J_L(\vartheta)}} \inf_{m \in F} \mathscr{W}_{Lh}(m) + 2\vartheta\delta + 3\varepsilon + \frac{\log(3/\delta)}{\beta L} \tag{A.17}$$

holds true, providing that $L > L_0(\varepsilon)$. Choosing now, for instance, $\varepsilon = c/18$, $2\vartheta\delta \in (0, c/6]$, and L so large that $\frac{\log(3/\delta)}{\beta L} \leqslant c/6$ as well as $L > L_0(c/18)$, the lemma follows due to (A.7). ▌

Lemma A.6. Let $\beta > \beta_c$ and let $\{a_L\}$ and $\{b_L\}$ be two sequences of positive numbers smaller that $\Delta m(B^\star)/2$. Defining $f_B(m) := \mathscr{W}_0(m) - Bm$, $B \in \mathbb{R}$, and $A_L(B) := [-m^* + a_L, m_+(B) - b_L] \cup [m_+(B) + b_L, m^*]$, we have

$$\min_{m \in A_L(B)} f_B(m) = f_B(-m^* + a_L) \tag{A.18}$$

for $B \leqslant \tau/m^*$, whereas

$$\min_{m \in A_L(B)} f_B(m) \geqslant \min\{f_B(-m^* + a_L), f_B(m(B) - b_L), f_B(m(B) + b_L)\} \tag{A.19}$$

for $B \geqslant \tau/m^*$.

Proof. Let $\beta > \beta_c$ and let sequences $\{a_L\}$ and $\{b_L\}$ be given. Taking into account that $m_+ \geqslant -m^* + \epsilon_0$, it follows that $[-m^* + a_L, m_+(B) - b_L] \neq \emptyset$ since

$$m_+(B) - b_L - (-m^* + a_L) = m_+(B) - (-m^*) - (a_L + b_L) > 2\epsilon_0 > 0. \tag{A.20}$$

Notice, however, that, taking into account (2.6), one has $m_+(B) + b_L > m^*$ when $B > (\sqrt{\kappa/b_L})/2$.

We shall use the following properties of the the function f_B:

(a) it is strictly concave on $[-m^*, m_t]$,

(b) it is strictly convex on $[m_t, m^*]$,

(c) it is increasing on $[-m^*, m^*]$ for any $B \leqslant \tau/m^*$, and

(d) it has a local minimum at $m(B)$ for all $B > \tau/m^*$;

they all directly follow from (3.5). Recall that $m_t = -m^*(1 - \frac{w^2}{8\tau^2}) \in (-m^*, m^*)$.

Since a continuous and strictly concave function attains its minimum over an interval at the end-point(s) of the interval, the property (a) implies

$$\min_{m \in [-m^* + a_L, m_t]} f_B(m) = \min\{f_B(-m^* + a_L), f_B(m_t)\} \qquad \text{for all} \quad B \in \mathbb{R}. \tag{A.21}$$

Moreover, in view of (c), we have

$$\min_{m \in A_L(B)} f_B(m) = f_B(-m^* + a_L) \qquad \text{for all} \quad B \leqslant \tau/m^*. \tag{A.22}$$

With the help of (b) and (d), obviously,

$$\min_{m \in [m_t, m^*]} f_B(m) = \min_{m \in [m(B) - b_L, m(B) + b_L] \cap [m_t, m^*]} f_B(m) \qquad \text{for all} \quad B \geqslant \tau / m^*. \tag{A.23}$$

Using, further, the fact that $B^\star = \tau / m^*$ (see the remark after Theorem 3.2), in view of (2.7) we have $m_+(B) = m(B)$ whenever $B \geqslant \tau / m^*$ and thus

$$\min_{m \in A_L(B) \cap [m_t, m^*]} f_B(m) = \min_{m \in \{m(B) - b_L, m(B) + b_L\} \cap [m_t, m^*]} f_B(m) \qquad \text{for} \quad B \geqslant \tau / m^*. \tag{A.24}$$

Finally, observing that

$$\begin{aligned} \min_{m \in A_L(B)} f_B(m) &= \min\{ \min_{m \in A_L(B) \cap [-m^* + a_L, m_t]} = f_B(m), \min_{m \in A_L(B) \cap [m_t, m^*]} f_B(m)\} \\ &\geqslant \min\{ \min_{m \in [-m^* + a_L, m_t]} f_B(m), \min_{m \in A_L(B) \cap [m_t, m^*]} f_B(m)\}, \end{aligned} \tag{A.25}$$

the lemma follows from (A.21), (A.24), and (A.22). ▮

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